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Problem 1: Let ABC be triangle with $BC=a, CA=b, AB=c$. Prove that

$$\frac{\sqrt{(a+b-c)^2+(a-b+c)^2}}{(b+c)^2}+\frac{\sqrt{(b+c-a)^2+(b-c+a)^2}}{(c+a)^2}+\frac{\sqrt{(c+a-b)^2+(c-a+b)^2}}{(a+b)^2}$$
$$\geq \frac{9\sqrt{2}}{4(a+b+c)}$$

Solution:

Using *AM-GM* inequality, we get

$$\sqrt{(a+b-c)^2+(a-b+c)^2} \geq \frac{a+b-c+a-b+c}{\sqrt{2}}=\sqrt{2}a$$

Deduce

$$\frac{\sqrt{(a+b-c)^2+(a-b+c)^2}}{(b+c)^2}+\frac{\sqrt{(b+c-a)^2+(b-c+a)^2}}{(c+a)^2}+\frac{\sqrt{(c+a-b)^2+(c-a+b)^2}}{(a+b)^2}$$
$$\geq \sqrt{2}\left(\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right)$$

By *Cauchy-Schwarz* inequality, we get

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \geq \frac{1}{a+b+c}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2 \geq \frac{9}{4(a+b+c)}$$

Thus

$$\frac{\sqrt{(a+b-c)^2+(a-b+c)^2}}{(b+c)^2}+\frac{\sqrt{(b+c-a)^2+(b-c+a)^2}}{(c+a)^2}+\frac{\sqrt{(c+a-b)^2+(c-a+b)^2}}{(a+b)^2}$$
$$\geq \frac{9\sqrt{2}}{4(a+b+c)}$$

Notes

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b^n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{a_1b_1+a_2b_2+\cdots+a_nb_n}$$

$$\frac{a_1}{b_1^2}+\frac{a_2}{b_2^2}+\cdots+\frac{a_n}{b_n^2} \geq \frac{1}{a_1+a_2+\cdots+a_n}\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_n}{b^n}\right)^2$$
$$\geq \frac{1}{a_1+a_2+\cdots+a_n}\frac{(a_1+a_2+\cdots+a_n)^4}{(a_1b_1+a_2b_2+\cdots+a_nb_n)^2}$$
$$=\frac{(a_1+a_2+\cdots+a_n)^3}{(a_1b_1+a_2b_2+\cdots+a_nb_n)^2}$$

Problem 2: Let ABC be triangle with $BC=a, CA=b, AB=c$. Prove that

$$\frac{(b+c)^2}{(a+b-c)(a-b+c)}+\frac{(c+a)^2}{(b+c-a)(b-c+a)}+\frac{(a+b)^2}{(c+a-b)(c-a+b)}\geq 12$$

Solution:

Using *AM-GM* inequality, we get

$$(a+b-c)(a-b+c)\leq a^2$$

As such

$$\frac{(b+c)^2}{(a+b-c)(a-b+c)}+\frac{(c+a)^2}{(b+c-a)(b-c+a)}+\frac{(a+b)^2}{(c+a-b)(c-a+b)}$$

$$\geq \frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}$$

By *Cauchy-Schwarz* inequality and *AM-GM* inequality, we get

$$\frac{(b+c)^2}{a^2}+\frac{(c+a)^2}{b^2}+\frac{(a+b)^2}{c^2}\geq \frac{1}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)^2=12$$

Thus

$$\frac{(b+c)^2}{(a+b-c)(a-b+c)}+\frac{(c+a)^2}{(b+c-a)(b-c+a)}+\frac{(a+b)^2}{(c+a-b)(c-a+b)}\geq 12$$

Problem 3: Let ABC be triangle with $BC=a, CA=b, AB=c$. Prove that

$$\frac{(b+c)^2}{\sqrt{(a+b-c)(a-b+c)}}+\frac{(c+a)^2}{\sqrt{(b+c-a)(b-c+a)}}+\frac{(a+b)^2}{\sqrt{(c+a-b)(c-a+b)}}\geq \frac{36abc}{ab+bc+ca}$$

Solution:

Using *AM-GM* inequality, we get

$$\sqrt{(a+b-c)(a-b+c)}\leq a$$

As such

$$\frac{(b+c)^2}{\sqrt{(a+b-c)(a-b+c)}}+\frac{(c+a)^2}{\sqrt{(b+c-a)(b-c+a)}}+\frac{(a+b)^2}{\sqrt{(c+a-b)(c-a+b)}}$$

$$\geq \frac{(b+c)^2}{a}+\frac{(c+a)^2}{b}+\frac{(a+b)^2}{c}$$

By *Cauchy-Schwarz*, we get

$$\frac{(b+c)^2}{a}+\frac{(c+a)^2}{b}+\frac{(a+b)^2}{c}\geq \frac{4(a+b+c)^2}{a+b+c}=4(a+b+c)$$

On the other hand

$$a+b+c\geq \frac{9}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{9abc}{ab+bc+ca}$$

Thus

$$\frac{(b+c)^2}{\sqrt{(a+b-c)(a-b+c)}}+\frac{(c+a)^2}{\sqrt{(b+c-a)(b-c+a)}}+\frac{(a+b)^2}{\sqrt{(c+a-b)(c-a+b)}}\ge\frac{36abc}{ab+bc+ca}$$

Problem 4: Let ABC be triangle with $BC=a, CA=b, AB=c$. Prove that

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)}+\frac{1}{(c+a)^2(b+c-a)(b-c+a)}+\frac{1}{(a+b)^2(c+a-b)(c-a+b)}$$
$$\ge\frac{9}{4\left(a^4+b^4+c^4\right)}$$

Solution:

Using *AM-GM* inequality, we get

$$(a+b-c)(a-b+c)\le a^2$$
$$(b+c)^2\le 2\left(b^2+c^2\right)$$

Deduce

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)}+\frac{1}{(c+a)^2(b+c-a)(b-c+a)}+\frac{1}{(a+b)^2(c+a-b)(c-a+b)}$$
$$\ge\frac{1}{2}\left(\frac{1}{a^2\left(b^2+c^2\right)}+\frac{1}{b^2\left(c^2+a^2\right)}+\frac{1}{c^2\left(a^2+b^2\right)}\right)$$

On the other hand, by *AM-GM* and *Cauchy-Schwarz* inequalities , we obtain

$$\frac{1}{a^2\left(b^2+c^2\right)}+\frac{1}{b^2\left(c^2+a^2\right)}+\frac{1}{c^2\left(a^2+b^2\right)}\ge\frac{9}{2\left(a^2b^2+b^2c^2+c^2a^2\right)}$$

Deduce

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)}+\frac{1}{(c+a)^2(b+c-a)(b-c+a)}+\frac{1}{(a+b)^2(c+a-b)(c-a+b)}$$
$$\ge\frac{9}{4\left(a^4+b^4+c^4\right)}$$

Problem 5: Let ABC be triangle with $BC=a, CA=b, AB=c$ and area S. Prove that

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)}+\frac{1}{(c+a)^2(b+c-a)(b-c+a)}+\frac{1}{(a+b)^2(c+a-b)(c-a+b)}\ge\left(\frac{3}{8S}\right)^2$$

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