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*Problem 1:* Let  $ABC$  be triangle with  $BC = a, CA = b, AB = c$ . Prove that

$$\begin{aligned} & \frac{\sqrt{(a+b-c)^2 + (a-b+c)^2}}{(b+c)^2} + \frac{\sqrt{(b+c-a)^2 + (b-c+a)^2}}{(c+a)^2} + \frac{\sqrt{(c+a-b)^2 + (c-a+b)^2}}{(a+b)^2} \\ & \geq \frac{9\sqrt{2}}{4(a+b+c)} \end{aligned}$$

*Solution:*

Using *AM-GM* inequality, we get

$$\sqrt{(a+b-c)^2 + (a-b+c)^2} \geq \frac{a+b-c+a-b+c}{\sqrt{2}} = \sqrt{2}a$$

Deduce

$$\begin{aligned} & \frac{\sqrt{(a+b-c)^2 + (a-b+c)^2}}{(b+c)^2} + \frac{\sqrt{(b+c-a)^2 + (b-c+a)^2}}{(c+a)^2} + \frac{\sqrt{(c+a-b)^2 + (c-a+b)^2}}{(a+b)^2} \\ & \geq \sqrt{2} \left( \frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \right) \end{aligned}$$

By *Cauchy-Schwarz* inequality, we get

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{1}{a+b+c} \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right)^2 \geq \frac{9}{4(a+b+c)}$$

Thus

$$\begin{aligned} & \frac{\sqrt{(a+b-c)^2 + (a-b+c)^2}}{(b+c)^2} + \frac{\sqrt{(b+c-a)^2 + (b-c+a)^2}}{(c+a)^2} + \frac{\sqrt{(c+a-b)^2 + (c-a+b)^2}}{(a+b)^2} \\ & \geq \frac{9\sqrt{2}}{4(a+b+c)} \end{aligned}$$

**Notes**

$$\begin{aligned} & \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{a_1b_1 + a_2b_2 + \dots + a_nb_n} \\ & \frac{a_1}{b_1^2} + \frac{a_2}{b_2^2} + \dots + \frac{a_n}{b_n^2} \geq \frac{1}{a_1 + a_2 + \dots + a_n} \left( \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \right)^2 \\ & \geq \frac{1}{a_1 + a_2 + \dots + a_n} \frac{(a_1 + a_2 + \dots + a_n)^4}{(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2} \\ & = \frac{(a_1 + a_2 + \dots + a_n)^3}{(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2} \end{aligned}$$

*Problem 2:* Let  $ABC$  be triangle with  $BC = a, CA = b, AB = c$ . Prove that

$$\frac{(b+c)^2}{(a+b-c)(a-b+c)} + \frac{(c+a)^2}{(b+c-a)(b-c+a)} + \frac{(a+b)^2}{(c+a-b)(c-a+b)} \geq 12$$

*Solution:*

Using *AM-GM* inequality, we get

$$(a+b-c)(a-b+c) \leq a^2$$

As such

$$\begin{aligned} & \frac{(b+c)^2}{(a+b-c)(a-b+c)} + \frac{(c+a)^2}{(b+c-a)(b-c+a)} + \frac{(a+b)^2}{(c+a-b)(c-a+b)} \\ & \geq \frac{(b+c)^2}{a^2} + \frac{(c+a)^2}{b^2} + \frac{(a+b)^2}{c^2} \end{aligned}$$

By *Cauchy-Schwarz* inequality and *AM-GM* inequality, we get

$$\frac{(b+c)^2}{a^2} + \frac{(c+a)^2}{b^2} + \frac{(a+b)^2}{c^2} \geq \frac{1}{3} \left( \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right)^2 = 12$$

Thus

$$\frac{(b+c)^2}{(a+b-c)(a-b+c)} + \frac{(c+a)^2}{(b+c-a)(b-c+a)} + \frac{(a+b)^2}{(c+a-b)(c-a+b)} \geq 12$$

*Problem 3:* Let  $ABC$  be triangle with  $BC = a, CA = b, AB = c$ . Prove that

$$\frac{(b+c)^2}{\sqrt{(a+b-c)(a-b+c)}} + \frac{(c+a)^2}{\sqrt{(b+c-a)(b-c+a)}} + \frac{(a+b)^2}{\sqrt{(c+a-b)(c-a+b)}} \geq \frac{36abc}{ab+bc+ca}$$

*Solution:*

Using *AM-GM* inequality, we get

$$\sqrt{(a+b-c)(a-b+c)} \leq a$$

As such

$$\begin{aligned} & \frac{(b+c)^2}{\sqrt{(a+b-c)(a-b+c)}} + \frac{(c+a)^2}{\sqrt{(b+c-a)(b-c+a)}} + \frac{(a+b)^2}{\sqrt{(c+a-b)(c-a+b)}} \\ & \geq \frac{(b+c)^2}{a} + \frac{(c+a)^2}{b} + \frac{(a+b)^2}{c} \end{aligned}$$

By *Cauchy-Schwarz*, we get

$$\frac{(b+c)^2}{a} + \frac{(c+a)^2}{b} + \frac{(a+b)^2}{c} \geq \frac{4(a+b+c)^2}{a+b+c} = 4(a+b+c)$$

On the other hand

$$a+b+c \geq \frac{9}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \frac{9abc}{ab+bc+ca}$$

Thus

$$\frac{(b+c)^2}{\sqrt{(a+b-c)(a-b+c)}} + \frac{(c+a)^2}{\sqrt{(b+c-a)(b-c+a)}} + \frac{(a+b)^2}{\sqrt{(c+a-b)(c-a+b)}} \geq \frac{36abc}{ab+bc+ca}$$

*Problem 4:* Let  $ABC$  be triangle with  $BC = a, CA = b, AB = c$ . Prove that

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)} + \frac{1}{(c+a)^2(b+c-a)(b-c+a)} + \frac{1}{(a+b)^2(c+a-b)(c-a+b)} \geq \frac{9}{4(a^4+b^4+c^4)}$$

*Solution:*

Using *AM-GM* inequality, we get

$$(a+b-c)(a-b+c) \leq a^2$$

$$(b+c)^2 \leq 2(b^2+c^2)$$

Deduce

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)} + \frac{1}{(c+a)^2(b+c-a)(b-c+a)} + \frac{1}{(a+b)^2(c+a-b)(c-a+b)} \geq \frac{1}{2} \left( \frac{1}{a^2(b^2+c^2)} + \frac{1}{b^2(c^2+a^2)} + \frac{1}{c^2(a^2+b^2)} \right)$$

On the other hand, by *AM-GM* and *Cauchy-Schwarz* inequalities, we obtain

$$\frac{1}{a^2(b^2+c^2)} + \frac{1}{b^2(c^2+a^2)} + \frac{1}{c^2(a^2+b^2)} \geq \frac{9}{2(a^2b^2+b^2c^2+c^2a^2)}$$

Deduce

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)} + \frac{1}{(c+a)^2(b+c-a)(b-c+a)} + \frac{1}{(a+b)^2(c+a-b)(c-a+b)} \geq \frac{9}{4(a^4+b^4+c^4)}$$

*Problem 5:* Let  $ABC$  be triangle with  $BC = a, CA = b, AB = c$  and area  $S$ . Prove that

$$\frac{1}{(b+c)^2(a+b-c)(a-b+c)} + \frac{1}{(c+a)^2(b+c-a)(b-c+a)} + \frac{1}{(a+b)^2(c+a-b)(c-a+b)} \geq \left( \frac{3}{8S} \right)^2$$

===== THE END =====