SOMES PROBLEM OF TRIANGLE INEQUALITY

Dao Hoang Viet, 61 Ly Thai To street, Pleiku city, Viet Nam country.

Problem 1: Let ABC be triangle. Prove that

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right)+2\tan\left(\frac{B}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right)+2\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right)+2\tan\left(\frac{A}{2}\right)}\geq\frac{1}{3}$$

Solution:

We have

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right)+2\tan\left(\frac{B}{2}\right)}=\frac{\tan^4\left(\frac{A}{2}\right)}{\tan^2\left(\frac{A}{2}\right)+2\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$$

Similarly

$$\frac{\tan^3\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right)+2\tan\left(\frac{C}{2}\right)}=\frac{\tan^4\left(\frac{B}{2}\right)}{\tan^2\left(\frac{B}{2}\right)+2\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

$$\frac{\tan^3\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right)+2\tan\left(\frac{A}{2}\right)}=\frac{\tan^4\left(\frac{C}{2}\right)}{\tan^2\left(\frac{C}{2}\right)+2\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)}$$

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right)+2\tan\left(\frac{B}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right)+2\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right)+2\tan\left(\frac{A}{2}\right)}$$

$$\geq\frac{\left(\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)\right)^2}{\left(\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{B}{2}\right)\right)^2}$$

On the other hand

$$\left(\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{B}{2}\right)\right)^2\leq3\left(\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)\right)$$

$$1=\left(\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)+\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)+\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)\right)^2$$

$$\leq\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)$$

Thus

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right)+2\tan\left(\frac{B}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right)+2\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right)+2\tan\left(\frac{A}{2}\right)}\geq\frac{1}{3}$$

Problem 2: Let ABC be triangle. Prove that

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right)+\tan\left(\frac{A}{2}\right)}\geq\frac{1}{2}$$

Problem 3: Let ABC be triangle with $BC=a, CA=b, AB=c$ and area S. Prove that

$$\frac{a^8}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}+\frac{b^8}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}+\frac{c^8}{\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}\geq\frac{4^5S^4}{3}$$

Solution:

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{a^8}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}+\frac{b^8}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}+\frac{c^8}{\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}$$

$$\geq\frac{\left(a^4+b^4+c^4\right)^2}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)+\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)+\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}$$

On the other hand

$$a^4+b^4+c^4\geq 16S^2$$

$$\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)+\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)+\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)\leq\frac{3}{4}$$

Thus

$$\frac{a^8}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}+\frac{b^8}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}+\frac{c^8}{\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}\geq\frac{4^5S^4}{3}$$

Problem 4: Let ABC be triangle with $BC=a, CA=b, AB=c$. Prove that

$$\frac{b^3}{c\cos^2\left(\frac{A}{2}\right)}+\frac{c^3}{a\cos^2\left(\frac{B}{2}\right)}+\frac{a^3}{b\cos^2\left(\frac{C}{2}\right)}\geq\frac{\left(a+b+c\right)^2}{36}$$

Solution:

We have

$$\frac{b^3}{c\cos^2\left(\frac{A}{2}\right)}=\frac{b^4}{bc\cos^2\left(\frac{A}{2}\right)},\frac{c^3}{a\cos^2\left(\frac{B}{2}\right)}=\frac{c^4}{ca\cos^2\left(\frac{B}{2}\right)},\frac{a^3}{b\cos^2\left(\frac{C}{2}\right)}=\frac{a^4}{ab\cos^2\left(\frac{C}{2}\right)}$$

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{b^3}{c\cos^2\left(\frac{A}{2}\right)}+\frac{c^3}{a\cos^2\left(\frac{B}{2}\right)}+\frac{a^3}{b\cos^2\left(\frac{C}{2}\right)}\geq\frac{\left(a^2+b^2+c^2\right)^2}{bc\cos^2\left(\frac{A}{2}\right)+ca\cos^2\left(\frac{B}{2}\right)+ab\cos^2\left(\frac{C}{2}\right)}$$

On the other hand

$$\left(a^2+b^2+c^2\right)^2\geq\frac{\left(a+b+c\right)^4}{9}$$

$$bc\cos^2\left(\frac{A}{2}\right)+ca\cos^2\left(\frac{B}{2}\right)+ab\cos^2\left(\frac{C}{2}\right)=\frac{\left(a+b+c\right)^2}{4}$$

Thus

$$\frac{b^3}{c\cos^2\left(\frac{A}{2}\right)}+\frac{c^3}{a\cos^2\left(\frac{B}{2}\right)}+\frac{a^3}{b\cos^2\left(\frac{C}{2}\right)}\geq\frac{\left(a+b+c\right)^2}{36}$$

Problem 5: Let ABC be triangle. Prove that

$$\frac{\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}+\frac{\cos\left(\frac{C+A}{2}\right)}{\cos\left(\frac{C-A}{2}\right)}+\frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}\geq\frac{3}{2}$$

Solution:

We have

$$\frac{\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}=\frac{2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B+C}{2}\right)}{2\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{B+C}{2}\right)}=\frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{B+C}{2}\right)}=\frac{\sin\left(A\right)}{\sin\left(B\right)+\sin\left(C\right)}$$

Similarly

$$\frac{\cos\left(\frac{C+A}{2}\right)}{\cos\left(\frac{C-A}{2}\right)}=\frac{\sin\left(B\right)}{\sin\left(C\right)+\sin\left(A\right)},\frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}=\frac{\sin\left(C\right)}{\sin\left(A\right)+\sin\left(B\right)}$$

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{\sin\left(A\right)}{\sin\left(B\right)+\sin\left(C\right)}+\frac{\sin\left(B\right)}{\sin\left(C\right)+\sin\left(A\right)}+\frac{\sin\left(C\right)}{\sin\left(A\right)+\sin\left(B\right)}\geq\frac{3}{2}$$

Thus

$$\frac{\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}+\frac{\cos\left(\frac{C+A}{2}\right)}{\cos\left(\frac{C-A}{2}\right)}+\frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}\geq\frac{3}{2}$$

Problem 6: Let ABC be triangle with $BC=a, CA=b, AB=c$. Prove that

$$\frac{a^3}{a^2+8bc}+\frac{b^3}{b^2+8ca}+\frac{c^3}{c^2+8ab}\geq\frac{a+b+c}{9}$$

Solution:

We have

$$\frac{a^3}{a^2+8bc}=\frac{a^4}{a^3+8abc},\frac{b^3}{b^2+8ca}=\frac{b^4}{b^3+8abc},\frac{c^3}{c^2+8ab}=\frac{c^4}{c^3+8abc}$$

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{a^3}{a^2+8bc}+\frac{b^3}{b^2+8ca}+\frac{c^3}{c^2+8ab}\geq\frac{\left(a^2+b^2+c^2\right)^2}{a^3+b^3+c^3+24abc}$$

On the other hand

$$\left(a^2+b^2+c^2\right)^2\geq\frac{(a+b+c)^4}{9}$$

$$a^3+b^3+c^3+24abc\leq(a+b+c)^3$$

As such

$$\frac{\left(a^2+b^2+c^2\right)^2}{a^3+b^3+c^3+24abc}\geq\frac{a+b+c}{9}$$

Thus

$$\frac{a^3}{a^2+8bc}+\frac{b^3}{b^2+8ca}+\frac{c^3}{c^2+8ab}\geq\frac{a+b+c}{9}$$

Problem 7: Let ABC be triangle and $n\in\mathbb{N}^*$. Prove that

$$\frac{\sqrt[n]{\sin A}+\sqrt[n]{\sin B}+\sqrt[n]{\sin C}}{\sqrt[n]{\cos\left(\frac{A}{2}\right)}+\sqrt[n]{\cos\left(\frac{B}{2}\right)}+\sqrt[n]{\cos\left(\frac{C}{2}\right)}}\leq 1$$

Solution:

Applying the inequality $\left(\frac{X+Y}{2}\right)^n\leq\frac{X^n+Y^n}{2};\forall A,B>0$, we obtain:

$$\left(\frac{\sqrt[n]{\sin A}+\sqrt[n]{\sin B}}{2}\right)^n\leq\frac{\sin A+\sin B}{2}=\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2}\leq\cos\left(\frac{C}{2}\right)$$

Deduce

$$\frac{\sqrt[n]{\sin A}+\sqrt[n]{\sin B}}{2}\leq\sqrt[n]{\cos\left(\frac{C}{2}\right)}$$

Similarly

$$\frac{\sqrt[n]{\sin B}+\sqrt[n]{\sin C}}{2}\leq\sqrt[n]{\cos\left(\frac{A}{2}\right)},\ \frac{\sqrt[n]{\sin C}+\sqrt[n]{\sin A}}{2}\leq\sqrt[n]{\cos\left(\frac{B}{2}\right)}$$

As such

$$\sqrt[n]{\sin A}+\sqrt[n]{\sin B}+\sqrt[n]{\sin C}\le\sqrt[n]{\cos\left(\frac{A}{2}\right)}+\sqrt[n]{\cos\left(\frac{B}{2}\right)}+\sqrt[n]{\cos\left(\frac{C}{2}\right)}$$

Thus

$$\frac{\sqrt[n]{\sin A}+\sqrt[n]{\sin B}+\sqrt[n]{\sin C}}{\sqrt[n]{\cos\left(\frac{A}{2}\right)}+\sqrt[n]{\cos\left(\frac{B}{2}\right)}+\sqrt[n]{\cos\left(\frac{C}{2}\right)}}\le 1$$

Problem 8: Let ABC be triangle and R is radius of the circle circumscribed of it. Called respectively r_a, r_b, r_c is the radius of circle escribed angle corresponding A, B, C . Prove that

$$\frac{\cos\left(\frac{A}{2}\right)}{r_a}+\frac{\cos\left(\frac{B}{2}\right)}{r_b}+\frac{\cos\left(\frac{C}{2}\right)}{r_c}\ge\frac{\sqrt{3}}{R}$$

Solution:

We have

$$r_a=4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$
$$r_b=4R\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$
$$r_c=4R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{\cos\left(\frac{A}{2}\right)}{r_a}+\frac{\cos\left(\frac{B}{2}\right)}{r_b}+\frac{\cos\left(\frac{C}{2}\right)}{r_c}\ge\frac{\left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right)^2}{4R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\left(\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\right)}$$

Applying the inequalities

$$\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\le\frac{3\sqrt{3}}{2}$$
$$\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\le\frac{3}{2}$$
$$\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\le\left(\frac{\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)}{3}\right)^3$$

Deduce

$$\frac{\left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right)^2}{4R\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\left(\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\right)}$$

$$\geq \frac{\left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right)^2}{4R\left(\frac{\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)}{3}\right)^3\left(\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\right)}$$

$$\geq \frac{27}{4R\left(\cos\left(\frac{A}{2}\right)+\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\right)\left(\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)\right)} \geq \frac{27}{4R.3\sqrt{3}.3} = \frac{\sqrt{3}}{R}$$

Thus

$$\frac{\cos\left(\frac{A}{2}\right)}{r_a}+\frac{\cos\left(\frac{B}{2}\right)}{r_b}+\frac{\cos\left(\frac{C}{2}\right)}{r_c} \geq \frac{\sqrt{3}}{R}$$

Problem 9: Let ABC be triangle with the radius of the circle circumscribed equal 1. Called respectively r_a, r_b, r_c is the radius of circle escribed angle corresponding A, B, C. Prove that

$$\frac{\cos\left(\frac{A}{2}\right)}{r_a}+\frac{\cos\left(\frac{B}{2}\right)}{r_b}+\frac{\cos\left(\frac{C}{2}\right)}{r_c} \geq \sqrt{3}$$

Problem 10: Let ABC be triangle. Prove that

$$\frac{\cot^4\left(\frac{A}{2}\right)}{\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}+\frac{\cot^4\left(\frac{B}{2}\right)}{\cot\left(\frac{C}{2}\right)+\cot\left(\frac{A}{2}\right)}+\frac{\cot^4\left(\frac{C}{2}\right)}{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)} \geq \frac{9\sqrt{3}}{2}$$

Solution:

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{\cot^4\left(\frac{A}{2}\right)}{\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}+\frac{\cot^4\left(\frac{B}{2}\right)}{\cot\left(\frac{C}{2}\right)+\cot\left(\frac{A}{2}\right)}+\frac{\cot^4\left(\frac{C}{2}\right)}{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)} \geq \frac{\left(\cot^2\left(\frac{A}{2}\right)+\cot^2\left(\frac{B}{2}\right)+\cot^2\left(\frac{C}{2}\right)\right)^2}{2\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)}$$

Applying $\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)=\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$ and the *AM-GM* inequality

$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)=\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)\le\left(\frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{3}\right)^3$$

Deduce

$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\ge 3\sqrt{3}$$

On the other hand

$$\left(\cot^2\left(\frac{A}{2}\right)+\cot^2\left(\frac{B}{2}\right)+\cot^2\left(\frac{C}{2}\right)\right)^2\ge\frac{\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)^4}{9}$$

As such

$$\frac{\left(\cot^2\left(\frac{A}{2}\right)+\cot^2\left(\frac{B}{2}\right)+\cot^2\left(\frac{C}{2}\right)\right)^2}{2\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)}\ge\frac{\frac{\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)^4}{9}}{2\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)}$$

$$\ge\frac{3\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)^4}{2\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)^3}=\frac{3}{2}\left(\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)\right)\ge\frac{9\sqrt{3}}{2}$$

Thus

$$\frac{\cot^4\left(\frac{A}{2}\right)}{\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}+\frac{\cot^4\left(\frac{B}{2}\right)}{\cot\left(\frac{C}{2}\right)+\cot\left(\frac{A}{2}\right)}+\frac{\cot^4\left(\frac{C}{2}\right)}{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)}\ge\frac{9\sqrt{3}}{2}$$

Problem 11: Let ABC be triangle with $BC=a,CA=b,AB=c$. Prove that

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan^2\left(\frac{A}{2}\right)+8\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan^2\left(\frac{B}{2}\right)+8\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan^2\left(\frac{C}{2}\right)+8\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}\ge 81\sqrt{3}$$

Solution:

By the *Cauchy-Schwarz* inequality, we obtain

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan^2\left(\frac{A}{2}\right)+8\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan^2\left(\frac{B}{2}\right)+8\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan^2\left(\frac{C}{2}\right)+8\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$$

$$\geq \frac{\left(\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)\right)^2}{\tan^3\left(\frac{A}{2}\right)+\tan^3\left(\frac{B}{2}\right)+\tan^3\left(\frac{C}{2}\right)+24\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

$$\geq \frac{\left(\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)+\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)+\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)\right)^2}{\tan^3\left(\frac{A}{2}\right)+\tan^3\left(\frac{B}{2}\right)+\tan^3\left(\frac{C}{2}\right)+24\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

$$= \frac{1}{\tan^3\left(\frac{A}{2}\right)+\tan^3\left(\frac{B}{2}\right)+\tan^3\left(\frac{C}{2}\right)+24\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

On the other hand

$$\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\leq \frac{1}{3\sqrt{3}}$$

$$\tan^3\left(\frac{A}{2}\right)+\tan^3\left(\frac{B}{2}\right)+\tan^3\left(\frac{C}{2}\right)+24\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)\leq \left(\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\right)^3$$

As such

$$\frac{\tan^3\left(\frac{A}{2}\right)}{\tan^2\left(\frac{A}{2}\right)+8\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}+\frac{\tan^3\left(\frac{B}{2}\right)}{\tan^2\left(\frac{B}{2}\right)+8\tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)}+\frac{\tan^3\left(\frac{C}{2}\right)}{\tan^2\left(\frac{C}{2}\right)+8\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}\geq 81\sqrt{3}$$

============================== THE END ==============================