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Problem 1: Give two number reals x, y nonequal zero and satisfy $(x+y)^3+4xy \geq 12$. Find minimum value of expression

$$4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+1\right)-\frac{1}{x^2}-\frac{1}{y^2}.$$

Solve:

Application inequality $(x+y)^2 \geq 4xy$ for assumption, we obtian:

$$12 \leq (x+y)^3+4xy \leq (x+y)^3+(x+y)^2$$
$$\Leftrightarrow (x+y)^3+(x+y)^2-12 \geq 0$$
$$\Leftrightarrow (x+y-2)\left((x+y)^2+3(x+y)+6\right) \geq 0$$
$$\Leftrightarrow x+y \geq 2.$$

Follow inequality $4x^2y^2 \leq \left(x^2+y^2\right)^2$, we have

$$4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+1\right)-\frac{1}{x^2}-\frac{1}{y^2}=\frac{4\left(x^4+y^4+x^2y^2\right)-\left(x^2+y^2\right)}{x^2y^2}$$

$$=\frac{4\left(x^4+y^4+2x^2y^2-x^2y^2\right)-\left(x^2+y^2\right)}{x^2y^2}=\frac{4\left(x^2+y^2\right)^2-4x^2y^2-\left(x^2+y^2\right)}{x^2y^2}$$

$$\geq \frac{4\left(x^2+y^2\right)^2-\left(x^2+y^2\right)^2-\left(x^2+y^2\right)}{x^2y^2} \geq \frac{12\left(x^2+y^2\right)^2-4\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}=12-\frac{4}{x^2+y^2}$$

$$\geq 12-4.\frac{2}{(x+y)^2}=10.$$

Hence $\min\left(4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+1\right)-\frac{1}{y^2}-\frac{1}{x^2}\right)=10.$

Sign equal happen when $x=y$ and $x^2+y^2=2$. Combine $x+y \geq 2$ deduce $x=y=1$.

Problem 2: Give two number reals x, y nonequal zero and satisfy $(x+y)^3+4xy \geq 12$. Prove that

$$4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+1\right) \geq \frac{1}{x^2}+\frac{1}{y^2}+10.$$