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APPLICATION MONOTONE OF FUNCTION
PROVE THE INEQUALITY

Problem 1: Let x be positive real numbers. Prove that

a) $x-\frac{x^2}{2}<\ln(1+x)<x$;

b) $\ln(1+x)>\frac{x}{x+1}$;

c) $e^x>1+\ln(1+x)$.

Solution:

a) We need prove $x-\frac{x^2}{2}<\ln(1+x)$

Consider the function $f(x)=x-\frac{x^2}{2}-\ln(1+x)$ with $x>0$.

We have

$$f'(x)=1-x-\frac{1}{1+x}=\frac{-x^2}{1+x}<0,\ \forall x>0$$

As such f *decreasing* over $(0;+\infty)$ and $\lim\limits_{x\to+\infty} f(x)=-\infty$

Thus $\forall x>0:\ f(x)<f(0)=0$ or $x-\frac{x^2}{2}<\ln(1+x),\ \forall x>0$.

Similarly, consider the function $g(x)=\ln(1+x)-x$ with $x>0$.

We have

$$g'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}<0,\ \forall x>0$$

As such g *decreasing* over $(0;+\infty)$ and $\lim\limits_{x\to+\infty} f(x)=-\infty$

Thus $\forall x>0:\ g(x)<g(0)=0$ or $\ln(1+x)<x,\ \forall x>0$.

b) Consider the function $h(x)=\ln(1+x)-\frac{x}{1+x}$ with $x>0$

We have

$$h'(x)=\frac{1}{1+x}-\frac{1}{(1+x)^2}=\frac{x}{(1+x)^2}>0,\ \forall x>0$$

As such h *increasing* over $(0;+\infty)$ and $\lim\limits_{x\to+\infty} f(x)=+\infty$

Thus $\forall x>0:\ h(x)>h(0)=0$ or $\ln(1+x)>\frac{x}{1+x},\ \forall x>0$.

c) Consider $f(x)=e^x-1-\ln(1+x)$ with $x>0$

We have

$$f'(x) = e^x - \frac{1}{1+x},\ f'(x) = 0 \Leftrightarrow x = 0$$

On the other hand

$$f''(x) = e^x + \frac{1}{(1+x)^2} > 0,\ \forall x > 0$$

As such f' *increasing* over $(0;+\infty)$ and $\lim\limits_{x\to+\infty} f'(x) = +\infty$. Deduce

$$f'(x) > f'(0) = 0$$

Thus f *increasing* over $(0;+\infty)$ and $\lim\limits_{x\to+\infty} f(x) = +\infty$ or

$$\forall x > 0:\ f(x) > f(0) = 0 \Leftrightarrow e^x > 1 + \ln(1+x),\ \forall x > 0.$$

Problem 2: Prove that

a) $\frac{\sin x}{x} > \frac{2}{\pi}$ all $x \in \left(0;\frac{\pi}{2}\right)$;

b) $x - \frac{x^3}{3} < \arcsin x$ all $x \in (0;1]$.

Solution

a) Consider the function $f(x) = \frac{\sin x}{x}$ with $x \in \left(0;\frac{\pi}{2}\right)$

Easy deduce $x - \tan x < 0,\ \forall x \in \left(0;\frac{\pi}{2}\right)$

We have

$$f'(x) = \frac{\cos x(x - \tan x)}{x^2} < 0,\ \forall x \in \left(0;\frac{\pi}{2}\right)$$

As such f *decreasing* over $\left(0;\frac{\pi}{2}\right)$ and $\lim\limits_{x\to0^+}\frac{\sin x}{x} = 1$.

Thus $\forall x \in \left(0;\frac{\pi}{2}\right):\ f(x) > f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Leftrightarrow \frac{\sin x}{x} > \frac{2}{\pi},\ \forall x \in \left(0;\frac{\pi}{2}\right)$

b) Consider the function $g(x) = x - \frac{x^3}{3} - \arcsin x$ with $x \in (0;1]$.

We have

$$g'(x) = 1 - x^2 - \frac{1}{\sqrt{1-x^2}} = \frac{(1-x^2)\sqrt{1-x^2} - 1}{\sqrt{1-x^2}},\ g'(x) = 0 \Leftrightarrow x = 0 \notin (0;1]$$

On the other hand

$$g''(x) = -2x - \frac{x}{(1-x^2)\sqrt{1-x^2}} < 0,\ \forall x \in (0;1) \text{ and } \lim_{x\to1^-} g''(x) = -\infty$$

As such g' *decreasing* over $(0;1]$ and $\lim\limits_{x\to1^-} g'(x) = -\infty$ or $g'(x) < g'(0) = 0$

Thus g *decreasing* over $(0;1]$ and $\lim\limits_{x\to1^+} g(x) = \frac{2}{3} - \frac{\pi}{2} < 0$ or

$$\forall x \in (0;1]:\ g(x) < g(0) = 0 \Leftrightarrow x - \frac{x^3}{3} < \arcsin x,\ \forall x \in (0;1]$$

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