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Problem 1: Let a,b,c be positive real numbers. Prove that

$$\frac{b^2(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{c^2(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{a^2(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}\ge\frac{2(a+b+c)\sqrt{a+b+c}}{3}$$

Solution:

We have

$$\frac{b^2(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{c^2(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{a^2(a+b)}{\sqrt{c\left(c^2+ca+ab\right)}}\ge\frac{2(a+b+c)\sqrt{a+b+c}}{3}$$

$$\Leftrightarrow\frac{1}{\sqrt{a+b+c}}\left(\frac{b^2(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{c^2(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{a^2(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}\right)\ge\frac{2(a+b+c)}{3}$$

By the *AM-GM* inequality, we obtain

$$\sqrt{a(a+b+c)\left(a^2+ab+bc\right)}\le\frac{a^2+ab+ac+a^2+ab+bc}{2}=\frac{(2a+c)(a+b)}{2}$$

Deduce

$$\frac{b^2(a+b)}{\sqrt{a(a+b+c)\left(a^2+ab+bc\right)}}\ge\frac{2b^2}{2a+c}$$

Similarly

$$\frac{c^2(b+c)}{\sqrt{b(a+b+c)\left(b^2+bc+ca\right)}}\ge\frac{2c^2}{2b+a}$$

$$\frac{a^2(c+a)}{\sqrt{c(a+b+c)\left(c^2+ca+ab\right)}}\ge\frac{2a^2}{2c+b}$$

Applying the *Cauchy-Schwarz* inequality, we have

$$\frac{2b^2}{2a+c}+\frac{2c^2}{2b+a}+\frac{2a^2}{2c+b}\ge\frac{2(a+b+c)^2}{3(a+b+c)}=\frac{2(a+b+c)}{3}$$

Thus

$$\frac{b^2(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{c^2(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{a^2(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}\ge\frac{2(a+b+c)\sqrt{a+b+c}}{3}$$

Problem 2: Let a,b,c be positive real numbers. Prove that

$$\frac{\left(a^2+2b^2\right)\left(b^2+2c^2\right)(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{\left(b^2+2c^2\right)\left(c^2+2a^2\right)(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{\left(c^2+2a^2\right)\left(a^2+2b^2\right)(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}$$

$$\ge\frac{2(a+b+c)^3\sqrt{a+b+c}}{3}$$

Solution:

We have

$$\frac{\left(a^2+2b^2\right)\left(b^2+2c^2\right)(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{\left(b^2+2c^2\right)\left(c^2+2a^2\right)(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{\left(c^2+2a^2\right)\left(a^2+2b^2\right)(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}$$

$$\geq \frac{2(a+b+c)^3\sqrt{a+b+c}}{3}$$

$$\Leftrightarrow \frac{1}{\sqrt{a+b+c}}\left(\frac{\left(a^2+2b^2\right)\left(b^2+2c^2\right)(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{\left(b^2+2c^2\right)\left(c^2+2a^2\right)(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{\left(c^2+2a^2\right)\left(a^2+2b^2\right)(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}\right)$$

$$\geq \frac{2(a+b+c)^3}{3}$$

By the *AM*-*GM* inequality, we obtain

$$\sqrt{a(a+b+c)\left(a^2+ab+bc\right)} \leq \frac{a^2+ab+ac+a^2+ab+bc}{2}=\frac{(2a+c)(a+b)}{2}$$

By the *Cauchy*-*Schwarz* inequality, we have

$$\left(a^2+2b^2\right)\left(b^2+2c^2\right)=\left(b^2+b^2+a^2\right)\left(b^2+c^2+c^2\right)\geq\left(b^2+bc+ca\right)^2$$

As such

$$\frac{\left(a^2+2b^2\right)\left(b^2+2c^2\right)(a+b)}{\sqrt{a(a+b+c)\left(a^2+ab+bc\right)}}\geq\frac{2\left(b^2+bc+ca\right)^2}{2a+c}$$

Similarly

$$\frac{\left(b^2+2c^2\right)\left(c^2+2a^2\right)(b+c)}{\sqrt{b(a+b+c)\left(b^2+bc+ca\right)}}\geq\frac{2\left(c^2+ca+ab\right)^2}{2b+a}$$

$$\frac{\left(c^2+2a^2\right)\left(a^2+2b^2\right)(c+a)}{\sqrt{c(a+b+c)\left(c^2+ca+ab\right)}}\geq\frac{2\left(a^2+ab+bc\right)^2}{2c+b}$$

Finally, by the *Cauchy*-*Schwarz* inequality, we obtain

$$\frac{2\left(b^2+bc+ca\right)^2}{2a+c}+\frac{2\left(c^2+ca+ab\right)^2}{2b+a}+\frac{2\left(a^2+ab+bc\right)^2}{2c+b}\geq\frac{2(a+b+c)^4}{3(a+b+c)}=\frac{2(a+b+c)^3}{3}$$

Thus

$$\frac{\left(a^2+2b^2\right)\left(b^2+2c^2\right)(a+b)}{\sqrt{a\left(a^2+ab+bc\right)}}+\frac{\left(b^2+2c^2\right)\left(c^2+2a^2\right)(b+c)}{\sqrt{b\left(b^2+bc+ca\right)}}+\frac{\left(c^2+2a^2\right)\left(a^2+2b^2\right)(c+a)}{\sqrt{c\left(c^2+ca+ab\right)}}$$

$$\geq \frac{2(a+b+c)^3\sqrt{a+b+c}}{3}$$

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