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Problem 1: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3b^3\left(a+b\right)}+\frac{1}{b^3c^3\left(b+c\right)}+\frac{1}{c^3a^3\left(c+a\right)}\ge\frac{3}{2}\left(\frac{3}{a+b+c}\right)^7.$$

Solution:

By *Hölder's* inequality, we get

$$\frac{1}{a^3b^3\left(a+b\right)}+\frac{1}{b^3c^3\left(b+c\right)}+\frac{1}{c^3a^3\left(c+a\right)}$$

$$=\frac{\frac{1}{a^3b^3}}{a+b}+\frac{\frac{1}{b^3c^3}}{b+c}+\frac{\frac{1}{c^3a^3}}{c+a}\ge\frac{\left(\frac{a+b+c}{abc}\right)^3}{6\left(a+b+c\right)}=\frac{\left(a+b+c\right)^2}{6\left(abc\right)^3}$$

Applying inequality

$$abc\le\left(\frac{a+b+c}{3}\right)^3$$

We have

$$\frac{\left(a+b+c\right)^2}{\left(abc\right)^3}\ge\frac{\left(a+b+c\right)^2}{\left(\frac{a+b+c}{3}\right)^9}=\frac{3^9}{\left(a+b+c\right)^7}$$

As such

$$\frac{1}{a^3b^3\left(a+b\right)}+\frac{1}{b^3c^3\left(b+c\right)}+\frac{1}{c^3a^3\left(c+a\right)}\ge\frac{3}{2}\left(\frac{3}{a+b+c}\right)^7.$$

Problem 2: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^2b^2\left(a+b\right)}+\frac{1}{b^2c^2\left(b+c\right)}+\frac{1}{c^2a^2\left(c+a\right)}\ge\frac{3}{2}\left(\frac{3}{a+b+c}\right)^5.$$

Solution:

By *Cauchy-Schwarz* inequality, we get

$$\frac{1}{a^2b^2\left(a+b\right)}+\frac{1}{b^2c^2\left(b+c\right)}+\frac{1}{c^2a^2\left(c+a\right)}$$

$$=\frac{\frac{1}{a^2b^2}}{a+b}+\frac{\frac{1}{b^2c^2}}{b+c}+\frac{\frac{1}{c^2a^2}}{c+a}\ge\frac{\left(\frac{a+b+c}{abc}\right)^2}{2\left(a+b+c\right)}=\frac{a+b+c}{2\left(abc\right)^2}$$

Applying inequality

$$abc\le\left(\frac{a+b+c}{3}\right)^3$$

We have

$$\frac{a+b+c}{(abc)^2} \ge \frac{a+b+c}{\left(\frac{a+b+c}{3}\right)^6} = \frac{3^6}{(a+b+c)^5}$$

As such

$$\frac{1}{a^2b^2(a+b)} + \frac{1}{b^2c^2(b+c)} + \frac{1}{c^2a^2(c+a)} \ge \frac{3}{2}\left(\frac{3}{a+b+c}\right)^5.$$

Problem 3: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^4b^4(a+b)} + \frac{1}{b^4c^4(b+c)} + \frac{1}{c^4a^4(c+a)} \ge \frac{3}{2}\left(\frac{3}{a+b+c}\right)^9.$$

Solution:

By *Cauchy-Schwarz* inequality, we obtain

$$\frac{1}{a^4b^4(a+b)} + \frac{1}{b^4c^4(b+c)} + \frac{1}{c^4a^4(c+a)}$$

$$= \frac{\left(\frac{1}{a^2b^2}\right)^2}{a+b} + \frac{\left(\frac{1}{b^2c^2}\right)^2}{b+c} + \frac{\left(\frac{1}{c^2a^2}\right)^2}{c+a} \ge \frac{\left(\frac{a^2+b^2+c^2}{a^2b^2c^2}\right)^2}{2(a+b+c)} = \frac{\left(a^2+b^2+c^2\right)^2}{2(abc)^4(a+b+c)}$$

Applying inequalities

$$abc \le \left(\frac{a+b+c}{3}\right)^3$$

$$a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}$$

We have

$$\frac{\left(a^2+b^2+c^2\right)^2}{(abc)^4(a+b+c)} \ge \frac{\frac{(a+b+c)^4}{9}}{\left(\frac{a+b+c}{3}\right)^{12}(a+b+c)} = \frac{3^{10}}{(a+b+c)^9}$$

As such

$$\frac{1}{a^4b^4(a+b)} + \frac{1}{b^4c^4(b+c)} + \frac{1}{c^4a^4(c+a)} \ge \frac{3}{2}\left(\frac{3}{a+b+c}\right)^9.$$

Problem 4: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b^2c^2(a+b)(a+c)} + \frac{1}{c^2a^2(b+c)(b+a)} + \frac{1}{a^2b^2(c+a)(c+b)} \ge \frac{3}{4}\left(\frac{3}{a+b+c}\right)^6.$$

Solution:

Using *Cauchy-Schwarz* inequality, we obtain

$$\frac{1}{a^2b^2(a+b)(a+c)}+\frac{1}{b^2c^2(b+c)(b+a)}+\frac{1}{c^2a^2(c+a)(c+b)}$$

$$=\frac{\frac{1}{a^2b^2}}{(a+b)(a+c)}+\frac{\frac{1}{b^2c^2}}{(b+c)(b+a)}+\frac{\frac{1}{c^2a^2}}{(c+a)(c+b)}$$

$$\geq\frac{\left(\frac{a+b+c}{abc}\right)^2}{\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}=\frac{(a+b+c)^2}{(abc)^2\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}$$

Applying inequalities

$$abc\leq\left(\frac{a+b+c}{3}\right)^3$$

$$a^2+b^2+c^2+3(ab+bc+ca)\leq\frac{4}{3}(a+b+c)^2$$

We have

$$\frac{(a+b+c)^2}{(abc)^2\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}\geq\frac{(a+b+c)^2}{\frac{4}{3}\left(\frac{a+b+c}{3}\right)^6(a+b+c)^2}$$

$$=\frac{3}{4}\left(\frac{3}{a+b+c}\right)^6$$

As such

$$\frac{1}{b^2c^2(a+b)(a+c)}+\frac{1}{c^2a^2(b+c)(b+a)}+\frac{1}{a^2b^2(c+a)(c+b)}\geq\frac{3}{4}\left(\frac{3}{a+b+c}\right)^6.$$

Problem 5: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b^4c^4(a+b)(a+c)}+\frac{1}{c^4a^4(b+c)(b+a)}+\frac{1}{a^4b^4(c+a)(c+b)}\geq\frac{3}{4}\left(\frac{3}{a+b+c}\right)^{10}.$$

Solution:

By *Cauchy-Schwarz* inequality, we obtain

$$\frac{1}{b^4c^4(a+b)(a+c)}+\frac{1}{c^4a^4(b+c)(b+a)}+\frac{1}{a^4b^4(c+a)(c+b)}$$

$$=\frac{\left(\frac{1}{a^2b^2}\right)^2}{(a+b)(a+c)}+\frac{\left(\frac{1}{b^2c^2}\right)^2}{(b+c)(b+a)}+\frac{\left(\frac{1}{c^2a^2}\right)^2}{(c+a)(c+b)}$$

$$\geq\frac{\left(\frac{a^2+b^2+c^2}{a^2b^2c^2}\right)^2}{\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}=\frac{\left(a^2+b^2+c^2\right)^2}{(abc)^4\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}$$

Applying inequalities

$$abc \le \left(\frac{a+b+c}{3}\right)^3$$

$$a^2+b^2+c^2+3(ab+bc+ca) \le \frac{4}{3}(a+b+c)^2$$

$$a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}$$

We have

$$\frac{\left(a^2+b^2+c^2\right)^2}{(abc)^4\left(a^2+b^2+c^2+3(ab+bc+ca)\right)} \ge \frac{\frac{(a+b+c)^4}{9}}{\frac{4}{3}\left(\frac{a+b+c}{3}\right)^{12}(a+b+c)^2}$$

$$= \frac{3}{4}\left(\frac{3}{a+b+c}\right)^{10}$$

As such

$$\frac{1}{b^4c^4(a+b)(a+c)}+\frac{1}{c^4a^4(b+c)(b+a)}+\frac{1}{a^4b^4(c+a)(c+b)} \ge \frac{3}{4}\left(\frac{3}{a+b+c}\right)^{10}.$$

Problem 6: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{b^3c^3(a+b)(a+c)}+\frac{1}{c^3a^3(b+c)(b+a)}+\frac{1}{a^3b^3(c+a)(c+b)} \ge \frac{3}{4}\left(\frac{3}{a+b+c}\right)^{8}.$$

Solution:

By *Hölder's* inequality, we get

$$\frac{1}{b^3c^3(a+b)(a+c)}+\frac{1}{c^3a^3(b+c)(b+a)}+\frac{1}{a^3b^3(c+a)(c+b)}$$

$$= \frac{\frac{1}{a^3b^3}}{(a+b)(a+c)}+\frac{\frac{1}{b^3c^3}}{(b+c)(b+a)}+\frac{\frac{1}{c^3a^3}}{(c+a)(c+b)}$$

$$\ge \frac{\left(\frac{a+b+c}{abc}\right)^3}{3\left(a^2+b^2+c^2+3(ab+bc+ca)\right)} = \frac{(a+b+c)^3}{3(abc)^3\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}$$

Applying inequalities

$$abc \le \left(\frac{a+b+c}{3}\right)^3$$

$$a^2+b^2+c^2+3(ab+bc+ca) \le \frac{4}{3}(a+b+c)^2$$

We have

$$\frac{(a+b+c)^3}{(abc)^3\left(a^2+b^2+c^2+3(ab+bc+ca)\right)} \geq \frac{(a+b+c)^3}{\frac{4}{3}\left(\frac{a+b+c}{3}\right)^9 (a+b+c)^2}$$

$$= \frac{9}{4}\left(\frac{3}{a+b+c}\right)^8$$

As such

$$\frac{1}{b^3c^3(a+b)(a+c)} + \frac{1}{c^3a^3(b+c)(b+a)} + \frac{1}{a^3b^3(c+a)(c+b)} \geq \frac{3}{4}\left(\frac{3}{a+b+c}\right)^8.$$

Problem 7: Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1^3x_2^3\ldots x_{n-1}^3(x_1+x_2+\cdots+x_{n-1})} + \frac{1}{x_2^3x_3^3\ldots x_n^3(x_2+x_3+\cdots+x_n)} + \cdots$$
$$+ \frac{1}{x_n^3x_1^3\ldots x_{n-2}^3(x_n+x_1+\cdots+x_{n-2})} \geq \frac{n^2}{3(n-1)}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-2}.$$

Solution:

Using *Hölder's* inequality, we obtain

$$\frac{1}{x_1^3x_2^3\ldots x_{n-1}^3(x_1+x_2+\cdots+x_{n-1})} + \frac{1}{x_2^3x_3^3\ldots x_n^3(x_2+x_3+\cdots+x_n)} + \cdots$$
$$+ \frac{1}{x_n^3x_1^3\ldots x_{n-2}^3(x_n+x_1+\cdots+x_{n-2})}$$
$$= \frac{\frac{1}{x_1^3x_2^3\ldots x_{n-1}^3}}{x_1+x_2+\cdots+x_{n-1}} + \frac{\frac{1}{x_2^3x_3^3\ldots x_n^3}}{x_2+x_3+\cdots+x_n} + \cdots + \frac{\frac{1}{x_n^3x_1^3\ldots x_{n-2}^3}}{x_n+x_1+\cdots+x_{n-2}}$$
$$\geq \frac{\left(\frac{x_1+x_2+\cdots+x_n}{x_1x_2\cdots x_n}\right)^3}{3(n-1)(x_1+x_2+\cdots+x_n)} = \frac{(x_1+x_2+\cdots+x_n)^2}{3(n-1)(x_1x_2\cdots x_n)^3}$$

Applying inequality

$$x_1x_2\cdots x_n \leq \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

We have

$$\frac{(x_1+x_2+\cdots+x_n)^2}{(x_1x_2\cdots x_n)^3} \geq \frac{(x_1+x_2+\cdots+x_n)^2}{\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^{3n}} = \frac{n^{3n}}{(x_1+x_2+\cdots+x_n)^{3n-2}}$$

As such

$$\frac{1}{x_1^3x_2^3\ldots x_{n-1}^3(x_1+x_2+\cdots+x_{n-1})} + \frac{1}{x_2^3x_3^3\ldots x_n^3(x_2+x_3+\cdots+x_n)} + \cdots$$
$$+ \frac{1}{x_n^3x_1^3\ldots x_{n-2}^3(x_n+x_1+\cdots+x_{n-2})} \geq \frac{n^2}{3(n-1)}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-2}$$

Problem 8: Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1^2 x_2^2 \cdots x_{n-1}^2 (x_1 + x_2 + \cdots + x_{n-1})} + \frac{1}{x_2^2 x_3^2 \cdots x_n^2 (x_2 + x_3 + \cdots + x_n)} + \cdots + \frac{1}{x_n^2 x_1^2 \cdots x_{n-2}^2 (x_n + x_1 + \cdots + x_{n-2})} \ge \frac{n}{n-1}\left(\frac{n}{x_1 + x_2 + \cdots + x_n}\right)^{2n-1}.$$

Solution:

Using *Cauchy-Schwarz* inequality, we get

$$\frac{1}{x_1^2 x_2^2 ... x_{n-1}^2 (x_1 + x_2 + \cdots + x_{n-1})} + \frac{1}{x_2^2 x_3^2 ... x_n^2 (x_2 + x_3 + \cdots + x_n)} + \cdots + \frac{1}{x_n^2 x_1^2 ... x_{n-2}^2 (x_n + x_1 + \cdots + x_{n-2})}$$

$$= \frac{\frac{1}{x_1^2 x_2^2 ... x_{n-1}^2}}{x_1 + x_2 + \cdots + x_{n-1}} + \frac{\frac{1}{x_2^2 x_3^2 ... x_n^2}}{x_2 + x_3 + \cdots + x_n} + \cdots + \frac{\frac{1}{x_n^2 x_1^2 ... x_{n-2}^2}}{x_n + x_1 + \cdots + x_{n-2}}$$

$$\ge \frac{\left(\frac{x_1 + x_2 + \cdots + x_n}{x_1 x_2 \cdots x_n}\right)^2}{(n-1)(x_1 + x_2 + \cdots + x_n)} = \frac{x_1 + x_2 + \cdots + x_n}{(n-1)(x_1 x_2 \cdots x_n)^2}$$

Applying inequality

$$x_1 x_2 \cdots x_n \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n$$

We have

$$\frac{x_1 + x_2 + \cdots + x_n}{(x_1 x_2 \cdots x_n)^2} \ge \frac{x_1 + x_2 + \cdots + x_n}{\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^{2n}} = \frac{n^{2n}}{(x_1 + x_2 + \cdots + x_n)^{2n-1}}$$

As such

$$\frac{1}{x_1^2 x_2^2 \cdots x_{n-1}^2 (x_1 + x_2 + \cdots + x_{n-1})} + \frac{1}{x_2^2 x_3^2 \cdots x_n^2 (x_2 + x_3 + \cdots + x_n)} + \cdots + \frac{1}{x_n^2 x_1^2 \cdots x_{n-2}^2 (x_n + x_1 + \cdots + x_{n-2})} \ge \frac{n}{n-1}\left(\frac{n}{x_1 + x_2 + \cdots + x_n}\right)^{2n-1}$$

Problem 9: Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1^4 x_2^4 \cdots x_{n-1}^4 (x_1 + x_2 + \cdots + x_{n-1})} + \frac{1}{x_2^4 x_3^4 \cdots x_n^4 (x_2 + x_3 + \cdots + x_n)} + \cdots + \frac{1}{x_n^4 x_1^4 \cdots x_{n-2}^4 (x_n + x_1 + \cdots + x_{n-2})} \ge \frac{n}{n-1}\left(\frac{n}{x_1 + x_2 + \cdots + x_n}\right)^{4n-3}$$

Solution:

By *Cauchy-Schwarz* inequality, we obtain

$$\frac{1}{x_1^4x_2^4\cdots x_{n-1}^4\left(x_1+x_2+\cdots+x_{n-1}\right)}+\frac{1}{x_2^4x_3^4\cdots x_n^4\left(x_2+x_3+\cdots+x_n\right)}+\cdots$$

$$+\frac{1}{x_n^4x_1^4\cdots x_{n-2}^4\left(x_n+x_1+\cdots+x_{n-2}\right)}$$

$$=\frac{\frac{1}{x_1^4x_2^4...x_{n-1}^4}}{x_1+x_2+\cdots+x_{n-1}}+\frac{\frac{1}{x_2^4x_3^4...x_n^4}}{x_2+x_3+\cdots+x_n}+\cdots+\frac{\frac{1}{x_n^4x_1^4...x_{n-2}^4}}{x_n+x_1+\cdots+x_{n-2}}$$

$$\geq\frac{\left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{x_1^2x_2^2\cdots x_n^2}\right)^2}{(n-1)\left(x_1+x_2+\cdots+x_n\right)}=\frac{\left(x_1^2+x_2^2+\cdots+x_n^2\right)^2}{(n-1)\left(x_1x_2\cdots x_n\right)^4\left(x_1+x_2+\cdots+x_n\right)}$$

Applying inequalities

$$x_1x_2\cdots x_n\leq\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

$$x_1^2+x_2^2+\cdots+x_n^2\geq\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{n}$$

We have

$$\frac{\left(x_1^2+x_2^2+\cdots+x_n^2\right)^2}{\left(x_1x_2\cdots x_n\right)^4\left(x_1+x_2+\cdots+x_n\right)}\geq\frac{\frac{\left(x_1+x_2+\cdots+x_n\right)^4}{n^2}}{\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^{4n}\left(x_1+x_2+\cdots+x_n\right)}$$

$$=\frac{n^{4n-2}}{\left(x_1+x_2+\cdots+x_n\right)^{4n-3}}$$

As such

$$\frac{1}{x_1^4x_2^4\cdots x_{n-1}^4\left(x_1+x_2+\cdots+x_{n-1}\right)}+\frac{1}{x_2^4x_3^4\cdots x_n^4\left(x_2+x_3+\cdots+x_n\right)}+\cdots$$

$$+\frac{1}{x_n^4x_1^4\cdots x_{n-2}^4\left(x_n+x_1+\cdots+x_{n-2}\right)}\geq\frac{n}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{4n-3}$$

Problem 10: Let $x_1,\ x_2,\ldots,\ x_n$ be real numbers and satisfy the condition

$$x_1+x_2+\cdots+x_n=0.$$

Prove that

$$\frac{16^{x_1}}{4^{x_1}+2^{x_2+x_3}}+\frac{16^{x_2}}{4^{x_2}+2^{x_3+x_4}}+\cdots+\frac{16^{x_n}}{4^{x_n}+2^{x_1+x_2}}\geq\frac{n}{2}.$$

Solution:

By *Cauchy-Schwarz* inequality, we obtain

$$\frac{16^{x_1}}{4^{x_1}+2^{x_2+x_3}}+\frac{16^{x_2}}{4^{x_2}+2^{x_3+x_4}}+\cdots+\frac{16^{x_n}}{4^{x_n}+2^{x_1+x_2}}$$
$$=\frac{\left(4^{x_1}\right)^2}{4^{x_1}+2^{x_2+x_3}}+\frac{\left(4^{x_2}\right)^2}{4^{x_2}+2^{x_3+x_4}}+\cdots+\frac{\left(4^{x_n}\right)^2}{4^{x_n}+2^{x_1+x_2}}$$
$$\geq\frac{\left(4^{x_1}+4^{x_2}+\cdots+4^{x_n}\right)^2}{4^{x_1}+4^{x_2}+\cdots+4^{x_n}+2^{x_2+x_3}+2^{x_3+x_4}+\cdots+2^{x_1+x_2}}$$
$$\geq\frac{\left(4^{x_1}+4^{x_2}+\cdots+4^{x_n}\right)^2}{2\left(4^{x_1}+4^{x_2}+\cdots+4^{x_n}\right)}=\frac{4^{x_1}+4^{x_2}+\cdots+4^{x_n}}{2}$$

Using *AM-GM* inequality, we obtain

$$4^{x_1}+4^{x_2}+\cdots+4^{x_n}\geq n\sqrt[n]{4^{x_1}4^{x_2}\cdots4^{x_n}}=n\sqrt[n]{4^{x_1+x_2+\cdots+x_n}}=n.$$

Thus

$$\frac{16^{x_1}}{4^{x_1}+2^{x_2+x_3}}+\frac{16^{x_2}}{4^{x_2}+2^{x_3+x_4}}+\cdots+\frac{16^{x_n}}{4^{x_n}+2^{x_1+x_2}}\geq\frac{n}{2}.$$

Problem 11: Let $x_1, x_2,\ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1x_2^2\cdots x_{n-1}^2x_n^2\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^2x_2x_3^2\cdots x_n^2\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^2\cdots x_{n-1}^2x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\geq\frac{2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{2n}$$

Solution:

By *Cauchy-Schwarz* inequality, we get

$$\frac{1}{x_1x_2^2\cdots x_{n-1}^2x_n^2\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^2x_2x_3^2\cdots x_n^2\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^2\cdots x_{n-1}^2x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}$$
$$=\frac{\frac{1}{x_2^2\cdots x_{n-1}^2x_n^2}}{x_1x_2+2x_1x_3\cdots+(n-1)x_1x_n}+\frac{\frac{1}{x_1^2x_3^2\cdots x_n^2}}{x_2x_3+2x_2x_4+\cdots+(n-1)x_2x_1}+\cdots$$
$$+\frac{\frac{1}{x_1^2\cdots x_{n-1}^2}}{x_nx_1+2x_nx_2\cdots+(n-1)x_nx_{n-1}}$$
$$\geq\frac{\left(\frac{x_1+x_2+\cdots+x_n}{x_1x_2\cdots x_n}\right)^2}{n\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$
$$=\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{n\left(x_1x_2\cdots x_n\right)^2\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$

Applying inequality

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+\cdots+\left(x_1-x_n\right)^2+\left(x_2-x_3\right)^2+\cdots+\left(x_2-x_n\right)^2+\cdots+\left(x_{n-1}-x_n\right)^2\ge 0$$

Deduce

$$x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\le\frac{n-1}{2n}\left(x_1+x_2+\cdots+x_n\right)^2$$

Applying *AM-GM* inequality

$$x_1x_2\cdots x_n\le\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

We have

$$\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{\left(x_1x_2\cdots x_n\right)^2\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$

$$\ge\frac{2n}{n-1}\frac{n^{2n}}{\left(x_1+x_2+\cdots+x_n\right)^{2n}}=\frac{2n}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{2n}$$

As such

$$\frac{1}{x_1x_2^2\cdots x_{n-1}^2x_n^2\left(x_2+2x_3\cdots+\left(n-1\right)x_n\right)}+\frac{1}{x_1^2x_2x_3^2\cdots x_n^2\left(x_3+2x_4+\cdots+\left(n-1\right)x_1\right)}+\cdots$$

$$+\frac{1}{x_1^2\cdots x_{n-1}^2x_n\left(x_1+2x_2\cdots+\left(n-1\right)x_{n-1}\right)}\ge\frac{2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{2n}$$

Problem 12: Let $x_1,\ x_2,\ldots,\ x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1x_2^4\cdots x_{n-1}^4x_n^4\left(x_2+2x_3\cdots+\left(n-1\right)x_n\right)}+\frac{1}{x_1^4x_2x_3^4\cdots x_n^4\left(x_3+2x_4+\cdots+\left(n-1\right)x_1\right)}+\cdots$$

$$+\frac{1}{x_1^4\cdots x_{n-1}^4x_n\left(x_1+2x_2\cdots+\left(n-1\right)x_{n-1}\right)}\ge\frac{2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{4n-2}$$

Solution:

By *Cauchy-Schwarz* inequality, we obtain

$$\frac{1}{x_1x_2^4\cdots x_{n-1}^4x_n^4\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^4x_2x_3^4\cdots x_n^4\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^4\cdots x_{n-1}^4x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}$$
$$=\frac{\frac{1}{x_2^4\cdots x_{n-1}^4x_n^4}}{x_1x_2+2x_1x_3\cdots+(n-1)x_1x_n}+\frac{\frac{1}{x_1^4x_3^4\cdots x_n^4}}{x_2x_3+2x_2x_4+\cdots+(n-1)x_2x_1}+\cdots$$
$$+\frac{\frac{1}{x_1^4\cdots x_{n-1}^4}}{x_nx_1+2x_nx_2\cdots+(n-1)x_nx_{n-1}}$$
$$\geq\frac{\left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{x_1x_2\cdots x_n}\right)^2}{n\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$
$$=\frac{\left(x_1^2+x_2^2+\cdots+x_n^2\right)^2}{n\left(x_1x_2\cdots x_n\right)^4\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$

Applying inequality

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+\cdots+\left(x_1-x_n\right)^2+\left(x_2-x_3\right)^2+\cdots+\left(x_2-x_n\right)^2+\cdots+\left(x_{n-1}-x_n\right)^2\geq 0$$

Deduce

$$x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\leq\frac{n-1}{2n}\left(x_1+x_2+\cdots+x_n\right)^2$$

Applying *AM-GM* inequality

$$x_1x_2\cdots x_n\leq\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

and $x_1^2+x_2^2+\cdots+x_n^2\geq\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{n}$ inequality, we have

$$\frac{\left(x_1^2+x_2^2+\cdots+x_n^2\right)^2}{n\left(x_1x_2\cdots x_n\right)^4\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$
$$\geq\frac{2n}{n-1}\frac{n^{4n}\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{n^2}}{\left(x_1+x_2+\cdots+x_n\right)^{4n}}=\frac{2n}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{4n-2}$$

As such

$$\frac{1}{x_1x_2^4\cdots x_{n-1}^4x_n^4\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^4x_2x_3^4\cdots x_n^4\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^4\cdots x_{n-1}^4x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\geq\frac{2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{4n-2}$$

Problem 13: Let $x_1, x_2,\ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1^3x_2^2\cdots x_{n-1}^2\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_2^3x_3^2\cdots x_n^2\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^2\cdots x_{n-2}^2x_n^3\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\geq\frac{2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{2n}$$

Solution:

By *Cauchy-Schwarz* inequality, we obtain

$$\frac{1}{x_1^3x_2^2\cdots x_{n-1}^2\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_2^3x_3^2\cdots x_n^2\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^2\cdots x_{n-2}^2x_n^3\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}$$
$$=\frac{\frac{1}{x_1^2x_2^2\cdots x_{n-1}^2}}{x_1x_2+2x_1x_3\cdots+(n-1)x_1x_n}+\frac{\frac{1}{x_2^2x_3^2\cdots x_n^2}}{x_2x_3+2x_2x_4+\cdots+(n-1)x_2x_1}+\cdots$$
$$+\frac{\frac{1}{x_1^2\cdots x_{n-2}^2x_n^2}}{x_nx_1+2x_nx_2\cdots+(n-1)x_nx_{n-1}}$$
$$\geq\frac{\left(\frac{x_1+x_2+\cdots+x_n}{x_1x_2\cdots x_n}\right)^2}{n\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$
$$=\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{n\left(x_1x_2\cdots x_n\right)^2\left(x_1x_2+x_1x_3+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+x_{n-1}x_n\right)}$$

Applying inequality

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+\cdots+\left(x_1-x_n\right)^2+\left(x_2-x_3\right)^2+\cdots+\left(x_2-x_n\right)^2+\cdots+\left(x_{n-1}-x_n\right)^2\geq 0$$

Deduce

$$x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\leq\frac{n-1}{2n}\left(x_1+x_2+\cdots+x_n\right)^2$$

Applying *AM-GM* inequality

$$x_1x_2\cdots x_n\leq\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

We have

$$\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{\left(x_1x_2\cdots x_n\right)^2\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$
$$\geq\frac{2n}{n-1}\frac{n^{2n}}{\left(x_1+x_2+\cdots+x_n\right)^{2n}}=\frac{2n}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{2n}$$

As such

$$\frac{1}{x_1^3x_2^2\cdots x_{n-1}^2\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_2^3x_3^2\cdots x_n^2\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^2\cdots x_{n-2}^2x_n^3\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\ge\frac{2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{2n}$$

Problem 14: Let $x_1, x_2,\ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1x_2^3\cdots x_{n-1}^3x_n^3\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^3x_2x_3^3\cdots x_n^3\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^3\cdots x_{n-1}^3x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\ge\frac{2n}{3(n-1)}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-1}$$

Solution:

By *Hölder's* inequality, we obtain

$$\frac{1}{x_1x_2^3\cdots x_{n-1}^3x_n^3\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^3x_2x_3^3\cdots x_n^3\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^3\cdots x_{n-2}^3x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}$$
$$=\frac{\frac{1}{x_2^3...x_{n-1}^3x_n^3}}{x_1x_2+2x_1x_3\cdots+(n-1)x_1x_n}+\frac{\frac{1}{x_1^3x_3^3...x_n^3}}{x_2x_3+2x_2x_4+\cdots+x_2x_n}+\cdots$$
$$+\frac{\frac{1}{x_1^3...x_{n-2}^3x_{n-1}^3}}{x_nx_1+2x_nx_2+\cdots+(n-1)x_nx_{n-1}}$$
$$\ge\frac{\left(\frac{x_1+x_2+\cdots+x_n}{x_1x_2\cdots x_n}\right)^3}{3n\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$
$$=\frac{\left(x_1+x_2+\cdots+x_n\right)^3}{3n\left(x_1x_2\cdots x_n\right)^3\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$

Applying inequality

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+\cdots+\left(x_1-x_n\right)^2+\left(x_2-x_3\right)^2+\cdots+\left(x_2-x_n\right)^2+\cdots+\left(x_{n-1}-x_n\right)^2\ge0$$

Deduce

$$x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\le\frac{n-1}{2n}\left(x_1+x_2+\cdots+x_n\right)^2$$

By *AM-GM* inequality

$$x_1x_2\cdots x_n\le\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

We have

$$\frac{\left(x_1+x_2+\cdots+x_n\right)^3}{\left(x_1x_2\cdots x_n\right)^3\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$
$$\geq \frac{2n}{n-1}\frac{n^{3n}}{\left(x_1+x_2+\cdots+x_n\right)^{3n-1}}=\frac{2n^2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-1}$$

As such

$$\frac{1}{x_1x_2^3\cdots x_{n-1}^3x_n^3\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_1^3x_2x_3^3\cdots x_n^3\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^3\cdots x_{n-1}^3x_n\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\geq\frac{2n}{3(n-1)}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-1}$$

Problem 15: Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{1}{x_1^4x_2^3\cdots x_{n-1}^3\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_2^4x_3^3\cdots x_n^3\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^3\cdots x_{n-2}^3x_n^4\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}\geq\frac{2n}{3(n-1)}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-1}.$$

Solution:

By *Hölder's* inequality, we obtain

$$\frac{1}{x_1^4x_2^3\cdots x_{n-1}^3\left(x_2+2x_3\cdots+(n-1)x_n\right)}+\frac{1}{x_2^4x_3^3\cdots x_n^3\left(x_3+2x_4+\cdots+(n-1)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^3\cdots x_{n-2}^3x_n^4\left(x_1+2x_2\cdots+(n-1)x_{n-1}\right)}$$
$$=\frac{\frac{1}{x_1^3x_2^3\ldots x_{n-1}^3}}{x_1x_2+2x_1x_3\cdots+(n-1)x_1x_n}+\frac{\frac{1}{x_2^3x_3^3\ldots x_n^3}}{x_2x_3+2x_2x_4+\cdots+x_2x_n}+\cdots$$
$$+\frac{\frac{1}{x_1^3\ldots x_{n-2}^3x_n^3}}{x_nx_1+2x_nx_2+\cdots+(n-1)x_nx_{n-1}}$$
$$\geq\frac{\left(\frac{x_1+x_2+\cdots+x_n}{x_1x_2\cdots x_n}\right)^3}{3n\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$
$$=\frac{\left(x_1+x_2+\cdots+x_n\right)^3}{3n\left(x_1x_2\cdots x_n\right)^3\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$

Using inequality

$$\left(x_1-x_2\right)^2+\left(x_1-x_3\right)^2+\cdots+\left(x_1-x_n\right)^2+\left(x_2-x_3\right)^2+\cdots+\left(x_2-x_n\right)^2+\cdots+\left(x_{n-1}-x_n\right)^2\geq 0$$

Deduce

$$x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\le\frac{n-1}{2n}\left(x_1+x_2+\cdots+x_n\right)^2$$

Applying *AM-GM* inequality

$$x_1x_2\cdots x_n\le\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

We have

$$\frac{\left(x_1+x_2+\cdots+x_n\right)^3}{\left(x_1x_2\cdots x_n\right)^3\left(x_1x_2+\cdots+x_1x_n+x_2x_3+\cdots+x_2x_n+\cdots+x_{n-1}x_n\right)}$$
$$\ge\frac{2n}{n-1}\frac{n^{3n}}{\left(x_1+x_2+\cdots+x_n\right)^{3n}}=\frac{2n^2}{n-1}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-1}$$

As such

$$\frac{1}{x_1^4x_2^3\cdots x_{n-1}^3\left(x_2+2x_3\cdots+\left(n-1\right)x_n\right)}+\frac{1}{x_2^4x_3^3\cdots x_n^3\left(x_3+2x_4+\cdots+\left(n-1\right)x_1\right)}+\cdots$$
$$+\frac{1}{x_1^3\cdots x_{n-2}^3x_n^4\left(x_1+2x_2\cdots+\left(n-1\right)x_{n-1}\right)}\ge\frac{2n}{3\left(n-1\right)}\left(\frac{n}{x_1+x_2+\cdots+x_n}\right)^{3n-1}$$

=========================THE END=====================