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Problem 1: Let $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2+x_3)^3}+\frac{x_2^4}{(x_3+x_4)^3}+\cdots+\frac{x_n^4}{(x_2+x_1)^3}\ge\frac{x_1+x_2+\cdots+x_n}{8}.$$

Solution:

We have

$$\frac{x_1^4}{(x_2+x_3)^3}=\frac{\frac{x_1^4}{(x_2+x_3)^2}}{x_2+x_3}$$

$$\frac{x_2^4}{(x_3+x_4)^3}=\frac{\frac{x_2^4}{(x_3+x_4)^2}}{x_3+x_4}$$

....

$$\frac{x_n^4}{(x_2+x_1)^3}=\frac{\frac{x_n^4}{(x_2+x_1)^2}}{x_2+x_1}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{x_1^4}{(x_2+x_3)^3}+\frac{x_2^4}{(x_3+x_4)^3}+\cdots+\frac{x_n^4}{(x_2+x_1)^3}\ge\frac{\left(\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_4}+\cdots+\frac{x_n^2}{x_2+x_1}\right)^2}{2(x_1+x_2+\cdots+x_n)}$$

On the other hand

$$\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_4}+\cdots+\frac{x_n^2}{x_2+x_1}\ge\frac{(x_1+x_2+\cdots+x_n)^2}{2(x_1+x_2+\cdots+x_n)}=\frac{x_1+x_2+\cdots+x_n}{2}$$

As such

$$\frac{x_1^4}{(x_2+x_3)^3}+\frac{x_2^4}{(x_3+x_4)^3}+\cdots+\frac{x_n^4}{(x_2+x_1)^3}\ge\frac{x_1+x_2+\cdots+x_n}{8}$$

Problem 2: Let $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that

$$\frac{x_1^3}{x_2x_3...x_n(x_2+x_3+\cdots+x_n)^3}+\frac{x_2^3}{x_1x_3...x_n(x_1+x_3+\cdots+x_n)^3}+\cdots$$
$$+\frac{x_n^3}{x_1x_2...x_{n-1}(x_1+x_2+\cdots+x_{n-1})^3}\ge\frac{n^n}{(n-1)^3(x_1+x_2+x_3+\cdots+x_n)^{n-1}}.$$

Solution:

We have

$$\frac{x_1^3}{x_2x_3...x_n\left(x_2+x_3+\cdots+x_n\right)^3}=\frac{\frac{x_1^4}{x_1x_2x_3...x_n\left(x_2+x_3+\cdots+x_n\right)^2}}{x_2+x_3+\cdots+x_n}$$

$$\frac{x_2^3}{x_1x_3...x_n\left(x_1+x_3+\cdots+x_n\right)^3}=\frac{\frac{x_2^4}{x_1x_2x_3...x_n\left(x_1+x_3+\cdots+x_n\right)^2}}{x_1+x_3+\cdots+x_n}$$

....

$$\frac{x_n^3}{x_1x_2...x_{n-1}\left(x_1+x_2+\cdots+x_{n-1}\right)^3}=\frac{\frac{x_n^4}{x_1x_2...x_{n-1}x_n\left(x_1+x_2+\cdots+x_{n-1}\right)^2}}{x_1+x_2+\cdots+x_{n-1}}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{x_1^3}{x_2x_3...x_n\left(x_2+x_3+\cdots+x_n\right)^3}+\frac{x_2^3}{x_1x_3...x_n\left(x_1+x_3+\cdots+x_n\right)^3}+\cdots$$

$$+\frac{x_n^3}{x_1x_2...x_{n-1}\left(x_1+x_2+\cdots+x_{n-1}\right)^3}$$

$$\geq\frac{\left(\frac{x_1^2}{x_2+x_3+\cdots+x_n}+\frac{x_2^2}{x_1+x_3+\cdots+x_n}+\cdots+\frac{x_n^2}{x_1+x_2+\cdots+x_{n-1}}\right)^2}{(n-1)x_1x_2x_3...x_n\left(x_1+x_2+x_3+\cdots+x_{n-1}+x_n\right)}$$

On the other hand

$$\frac{x_1^2}{x_2+x_3+\cdots+x_n}+\frac{x_2^2}{x_1+x_3+\cdots+x_n}+\cdots+\frac{x_n^2}{x_1+x_2+\cdots+x_{n-1}}$$

$$\geq\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{(n-1)\left(x_1+x_2+\cdots+x_n\right)}=\frac{x_1+x_2+\cdots+x_n}{n-1}$$

Applying the inequality *AM-GM*, we have

$$x_1x_2...x_n\leq\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n$$

Deduce

$$\frac{\left(\frac{x_1^2}{x_2+x_3+\cdots+x_n}+\frac{x_2^2}{x_1+x_3+\cdots+x_n}+\cdots+\frac{x_n^2}{x_1+x_2+\cdots+x_{n-1}}\right)^2}{(n-1)x_1x_2x_3...x_n\left(x_1+x_2+x_3+\cdots+x_{n-1}+x_n\right)}$$

$$\geq\frac{x_1+x_2+x_3+\cdots+x_n}{(n-1)^3\left(\frac{x_1+x_2+x_3+\cdots+x_{n-1}+x_n}{n}\right)^n}$$

$$=\frac{n^n}{(n-1)^3\left(x_1+x_2+x_3+\cdots+x_n\right)^{n-1}}.$$

As such

$$\frac{x_1^3}{x_2x_3...x_n\left(x_2+x_3+\cdots+x_n\right)^3}+\frac{x_2^3}{x_1x_3...x_n\left(x_1+x_3+\cdots+x_n\right)^3}+\cdots$$
$$+\frac{x_n^3}{x_1x_2...x_{n-1}\left(x_1+x_2+\cdots+x_{n-1}\right)^3}\geq\frac{n^n}{(n-1)^3\left(x_1+x_2+x_3+\cdots+x_n\right)^{n-1}}.$$

Problem 3: Let $x_1,\ x_2,\ldots,x_n$ be positive real numbers. Prove that

$$\frac{x_1^4}{\left(x_2+x_3\right)^4}+\frac{x_2^4}{\left(x_3+x_4\right)^4}+\cdots+\frac{x_n^4}{\left(x_2+x_1\right)^4}\geq\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{16\left(x_1^2+x_2^2+\cdots+x_n^2\right)}.$$

Solution:

We have

$$\frac{x_1^4}{\left(x_2+x_3\right)^4}=\frac{\dfrac{x_1^4}{\left(x_2+x_3\right)^2}}{\left(x_2+x_3\right)^2}$$
$$\frac{x_2^4}{\left(x_3+x_4\right)^4}=\frac{\dfrac{x_2^4}{\left(x_3+x_4\right)^2}}{\left(x_3+x_4\right)^2}$$
....
$$\frac{x_n^4}{\left(x_2+x_1\right)^4}=\frac{\dfrac{x_n^4}{\left(x_2+x_1\right)^2}}{\left(x_2+x_1\right)^2}$$

By the *Cauchy* – *Schwarz* inequality, we obtain

$$\frac{x_1^4}{\left(x_2+x_3\right)^4}+\frac{x_2^4}{\left(x_3+x_4\right)^4}+\cdots+\frac{x_n^4}{\left(x_2+x_1\right)^4}\geq\frac{\left(\dfrac{x_1^2}{x_2+x_3}+\dfrac{x_2^2}{x_3+x_4}+\cdots+\dfrac{x_n^2}{x_2+x_1}\right)^2}{\left(x_2+x_3\right)^2+\left(x_3+x_4\right)^2+\cdots+\left(x_2+x_1\right)^2}$$

On the other hand

$$\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_4}+\cdots+\frac{x_n^2}{x_2+x_1}\geq\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{2\left(x_1+x_2+\cdots+x_n\right)}=\frac{x_1+x_2+\cdots+x_n}{2}$$
$$\left(x_2+x_3\right)^2+\left(x_3+x_4\right)^2+\cdots+\left(x_2+x_1\right)^2\leq 4\left(x_1^2+x_2^2+\cdots+x_n^2\right)$$

Deduce

$$\frac{\left(\dfrac{x_1^2}{x_2+x_3}+\dfrac{x_2^2}{x_3+x_4}+\cdots+\dfrac{x_n^2}{x_2+x_1}\right)^2}{\left(x_2+x_3\right)^2+\left(x_3+x_4\right)^2+\cdots+\left(x_2+x_1\right)^2}\geq\frac{\dfrac{\left(x_1+x_2+\cdots+x_n\right)^2}{4}}{4\left(x_1^2+x_2^2+\cdots+x_n^2\right)}$$

As such

$$\frac{x_1^4}{\left(x_2+x_3\right)^4}+\frac{x_2^4}{\left(x_3+x_4\right)^4}+\cdots+\frac{x_n^4}{\left(x_2+x_1\right)^4}\geq\frac{\left(x_1+x_2+\cdots+x_n\right)^2}{16\left(x_1^2+x_2^2+\cdots+x_n^2\right)}$$

Problem 4: Let $x_1,\ x_2, x_3$ be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2+x_3)^4}+\frac{x_2^4}{(x_3+x_1)^4}+\frac{x_3^4}{(x_1+x_2)^4}\ge\frac{81x_1x_2x_3}{16(x_1+x_2+x_3)^3}.$$

Solution:

We have inequality

$$a^4+b^4+c^4\ge abc(a+b+c)\ \ \forall a,b,c>0$$

Thus

$$\frac{x_1^4}{(x_2+x_3)^4}+\frac{x_2^4}{(x_3+x_1)^4}+\frac{x_3^4}{(x_1+x_2)^4}$$

$$\ge\frac{x_1x_2x_3}{(x_2+x_3)(x_3+x_1)(x_1+x_2)}\left(\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_1}+\frac{x_3}{x_1+x_2}\right)$$

On the other hand

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_1}+\frac{x_3}{x_1+x_2}=\frac{x_1^2}{x_1(x_2+x_3)}+\frac{x_2^2}{x_2(x_3+x_1)}+\frac{x_3^2}{x_3(x_1+x_2)}$$

$$\ge\frac{(x_1+x_2+x_3)^2}{2(x_1x_2+x_2x_3+x_3x_1)}\ge\frac{3(x_1x_2+x_2x_3+x_3x_1)}{2(x_1x_2+x_2x_3+x_3x_1)}=\frac{3}{2}$$

$$\frac{x_1x_2x_3}{(x_2+x_3)(x_3+x_1)(x_1+x_2)}\ge\frac{x_1x_2x_3}{\left(\frac{x_2+x_3+x_3+x_1+x_1+x_2}{3}\right)^3}=\frac{27x_1x_2x_3}{8(x_1+x_2+x_3)^3}$$

As such

$$\frac{x_1^4}{(x_2+x_3)^4}+\frac{x_2^4}{(x_3+x_1)^4}+\frac{x_3^4}{(x_1+x_2)^4}\ge\frac{81x_1x_2x_3}{16(x_1+x_2+x_3)^3}$$

Problem 5: Let $x_1,\ x_2,\ldots,x_n$ be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2+x_3)^4}+\frac{x_2^4}{(x_3+x_1)^4}+\frac{x_3^4}{(x_1+x_2)^4}\ge\frac{(x_1+x_2+x_3)^2}{16(x_1^2+x_2^2+x_3^2)}.$$

Solution:

We have

$$\frac{x_1^4}{(x_2+x_3)^4}=\frac{\frac{x_1^4}{(x_2+x_3)^2}}{(x_2+x_3)^2}$$

$$\frac{x_2^4}{(x_3+x_1)^4}=\frac{\frac{x_2^4}{(x_3+x_1)^2}}{(x_3+x_1)^2}$$

$$\frac{x_3^4}{(x_1+x_2)^4}=\frac{\frac{x_3^4}{(x_1+x_2)^2}}{(x_1+x_2)^2}$$

By the *Cauchy* – *Schwarz* inequality, we obtain

$$\frac{x_1^4}{(x_2+x_3)^4}+\frac{x_2^4}{(x_3+x_1)^4}+\frac{x_3^4}{(x_1+x_2)^4}\ge\frac{\left(\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_1}+\frac{x_3^2}{x_1+x_2}\right)^2}{(x_2+x_3)^2+(x_3+x_1)^2+(x_1+x_2)^2}$$

On the other hand

$$\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_1}+\frac{x_3^2}{x_1+x_2}\ge\frac{(x_1+x_2+x_3)^2}{2(x_1+x_2+x_3)}=\frac{x_1+x_2+x_3}{2}$$

$$(x_2+x_3)^2+(x_3+x_1)^2+(x_1+x_2)^2\le 4\left(x_1^2+x_2^2+x_3^2\right)$$

Deduce

$$\frac{\left(\frac{x_1^2}{x_2+x_3}+\frac{x_2^2}{x_3+x_4}+\frac{x_3^2}{x_2+x_1}\right)^2}{(x_2+x_3)^2+(x_3+x_1)^2+(x_1+x_3)^2}\ge\frac{\frac{(x_1+x_2+x_3)^2}{4}}{4\left(x_1^2+x_2^2+x_3^2\right)}$$

As such

$$\frac{x_1^4}{(x_2+x_3)^4}+\frac{x_2^4}{(x_3+x_1)^4}+\frac{x_3^4}{(x_1+x_3)^4}\ge\frac{(x_1+x_2+x_3)^2}{16\left(x_1^2+x_2^2+x_3^2\right)}$$

Problem 6: Let $x_1,\ x_2,x_3$ be positive real numbers. Prove that

$$81x_1x_2x_3\left(x_1^2+x_2^2+x_3^2\right)\le(x_1+x_2+x_3)^5.$$

Problem 7: Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{bcd\,(b+c+d)^3}+\frac{b}{cda\,(c+d+a)^3}+\frac{c}{dab\,(d+a+b)^3}+\frac{d}{abc\,(a+b+c)^3}$$

$$\ge\frac{4}{27}\left(\frac{4}{a+b+c+d}\right)^5$$

Solution:

We have

$$\frac{a}{bcd\left(b+c+d\right)^3}=\frac{a^2}{abcd\left(b+c+d\right)^3},\ \frac{b}{cda\left(c+d+a\right)^3}=\frac{b^2}{abcd\left(c+d+a\right)^3}$$

$$\frac{c}{dab\left(d+a+b\right)^3}=\frac{c^2}{abcd\left(d+a+b\right)^3},\ \frac{d}{abc\left(a+b+c\right)^3}=\frac{d^2}{abcd\left(a+b+c\right)^3}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{a}{bcd\left(b+c+d\right)^3}+\frac{b}{cda\left(c+d+a\right)^3}+\frac{c}{dab\left(d+a+b\right)^3}+\frac{d}{abc\left(a+b+c\right)^3}$$

$$\geq\frac{\left(\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}\right)^2}{3abcd\left(a+b+c+d\right)}$$

On the other hand

$$\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}\geq\frac{4}{3}$$

Applying the inequality

$$abcd\leq\left(\frac{a+b+c+d}{4}\right)^4$$

As such

$$\frac{\left(\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}\right)^2}{abcd\left(a+b+c+d\right)}\geq\frac{\left(\frac{4}{3}\right)^2}{\left(\frac{a+b+c+d}{4}\right)^4\left(a+b+c+d\right)}$$

Thus

$$\frac{a}{bcd\left(b+c+d\right)^3}+\frac{b}{cda\left(c+d+a\right)^3}+\frac{c}{dab\left(d+a+b\right)^3}+\frac{d}{abc\left(a+b+c\right)^3}$$

$$\geq\frac{4}{27}\left(\frac{4}{a+b+c+d}\right)^5$$

Problem 8: Let a, b, c, d be positive real numbers. Prove that

$$\frac{1}{bcd\left(b+2c+3d\right)^3}+\frac{1}{cda\left(c+2d+3a\right)^3}+\frac{1}{dab\left(d+2a+3b\right)^3}+\frac{1}{abc\left(a+2b+3c\right)^3}$$

$$\geq\frac{1}{54}\left(\frac{4}{a+b+c+d}\right)^6$$

Solution:

We have

$$\frac{1}{bcd\left(b+2c+3d\right)^{3}}=\frac{\frac{a^{2}}{\left(b+2c+3d\right)^{2}}}{abcd\left(ab+2ac+3ad\right)},\ \frac{1}{cda\left(c+2d+3a\right)^{3}}=\frac{\frac{b^{2}}{\left(c+2d+3a\right)^{2}}}{abcd\left(bc+2bd+3ab\right)}$$

$$\frac{1}{dab\left(d+2a+3b\right)^{3}}=\frac{\frac{c^{2}}{\left(d+2a+3b\right)^{2}}}{abcd\left(cd+2ca+3bc\right)},\ \frac{1}{abc\left(a+2b+3c\right)^{3}}=\frac{\frac{d^{2}}{\left(a+2b+3c\right)^{2}}}{abcd\left(ad+2bd+3cd\right)}$$

By the *Cauchy* – *Schwarz* inequality, we obtain

$$\frac{1}{bcd\left(b+2c+3d\right)^{3}}+\frac{1}{cda\left(c+2d+3a\right)^{3}}+\frac{1}{dab\left(d+2a+3b\right)^{3}}+\frac{1}{abc\left(a+2b+3c\right)^{3}}$$

$$\geq\frac{\left(\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c}\right)^{2}}{4abcd\left(ab+ac+ad+bc+bd+cd\right)}$$

On the other hand

$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c}\geq\frac{2}{3}$$

Applying the inequality

$$abcd\leq\left(\frac{a+b+c+d}{4}\right)^{4}$$

$$ab+ac+ad+bc+bd+cd\leq\frac{3}{8}\left(a+b+c+d\right)^{2}$$

As such

$$\frac{\left(\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c}\right)^{2}}{abcd\left(ab+ac+ad+bc+bd+cd\right)}$$

$$\geq\frac{\left(\frac{2}{3}\right)^{2}}{\left(\frac{a+b+c+d}{4}\right)^{4}\frac{3}{8}\left(a+b+c+d\right)^{2}}$$

Thus

$$\frac{1}{bcd\left(b+2c+3d\right)^{3}}+\frac{1}{cda\left(c+2d+3a\right)^{3}}+\frac{1}{dab\left(d+2a+3b\right)^{3}}+\frac{1}{abc\left(a+2b+3c\right)^{3}}$$

$$\geq\frac{1}{54}\left(\frac{4}{a+b+c+d}\right)^{6}$$

Problem 9: Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{bcd\left(b+c\right)^{3}}+\frac{b}{cda\left(c+d\right)^{3}}+\frac{c}{dab\left(d+a\right)^{3}}+\frac{d}{abc\left(a+b\right)^{3}}$$

$$\geq\frac{1}{2}\left(\frac{4}{a+b+c+d}\right)^{5}$$

Solution:

We have

$$\frac{a}{bcd\left(b+c\right)^3}=\frac{\frac{a^2}{\left(b+c\right)^2}}{abcd\left(b+c\right)},\ \frac{b}{cda\left(c+d\right)^3}=\frac{\frac{b^2}{\left(c+d\right)^2}}{abcd\left(c+d\right)}$$

$$\frac{c}{dab\left(d+a\right)^3}=\frac{\frac{c^2}{\left(d+a\right)^2}}{abcd\left(d+a\right)},\ \frac{d}{abc\left(a+b\right)^3}=\frac{\frac{d^2}{\left(a+b\right)^2}}{abcd\left(a+b\right)}$$

By the *Cauchy* – *Schwarz* inequality, we obtain

$$\frac{a}{bcd\left(b+c\right)^3}+\frac{b}{cda\left(c+d\right)^3}+\frac{c}{dab\left(d+a\right)^3}+\frac{d}{abc\left(a+b\right)^3}$$

$$\geq\frac{\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)^2}{2abcd\left(a+b+c+d\right)}$$

On the other hand

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geq 2$$

Applying the inequality

$$abcd\leq\left(\frac{a+b+c+d}{4}\right)^4$$

As such

$$\frac{\left(\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\right)^2}{abcd\left(a+b+c+d\right)}\geq\frac{4}{\left(\frac{a+b+c+d}{4}\right)^4\left(a+b+c+d\right)}$$

Thus

$$\frac{a}{bcd\left(b+c\right)^3}+\frac{b}{cda\left(c+d\right)^3}+\frac{c}{dab\left(d+a\right)^3}+\frac{d}{abc\left(a+b\right)^3}$$

$$\geq\frac{1}{2}\left(\frac{4}{a+b+c+d}\right)^5$$

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