

Dao Hoang Viet, 61 Ly Thai To street, Pleiku city, Viet Nam country.

Problem 1: Let x_1, x_2, \dots, x_n be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2 + x_3)^3} + \frac{x_2^4}{(x_3 + x_4)^3} + \dots + \frac{x_n^4}{(x_2 + x_1)^3} \geq \frac{x_1 + x_2 + \dots + x_n}{8}.$$

Solution:

We have

$$\begin{aligned} \frac{x_1^4}{(x_2 + x_3)^3} &= \frac{\frac{x_1^4}{(x_2 + x_3)^2}}{x_2 + x_3} \\ \frac{x_2^4}{(x_3 + x_4)^3} &= \frac{\frac{x_2^4}{(x_3 + x_4)^2}}{x_3 + x_4} \\ &\dots\dots\dots \\ \frac{x_n^4}{(x_2 + x_1)^3} &= \frac{\frac{x_n^4}{(x_2 + x_1)^2}}{x_2 + x_1} \end{aligned}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{x_1^4}{(x_2 + x_3)^3} + \frac{x_2^4}{(x_3 + x_4)^3} + \dots + \frac{x_n^4}{(x_2 + x_1)^3} \geq \frac{\left(\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_4} + \dots + \frac{x_n^2}{x_2 + x_1} \right)^2}{2(x_1 + x_2 + \dots + x_n)}$$

On the other hand

$$\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_4} + \dots + \frac{x_n^2}{x_2 + x_1} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{2(x_1 + x_2 + \dots + x_n)} = \frac{x_1 + x_2 + \dots + x_n}{2}$$

As such

$$\frac{x_1^4}{(x_2 + x_3)^3} + \frac{x_2^4}{(x_3 + x_4)^3} + \dots + \frac{x_n^4}{(x_2 + x_1)^3} \geq \frac{x_1 + x_2 + \dots + x_n}{8}$$

Problem 2: Let x_1, x_2, \dots, x_n be positive real numbers. Prove that

$$\begin{aligned} &\frac{x_1^3}{x_2 x_3 \dots x_n (x_2 + x_3 + \dots + x_n)^3} + \frac{x_2^3}{x_1 x_3 \dots x_n (x_1 + x_3 + \dots + x_n)^3} + \dots \\ &+ \frac{x_n^3}{x_1 x_2 \dots x_{n-1} (x_1 + x_2 + \dots + x_{n-1})^3} \geq \frac{n^n}{(n-1)^3 (x_1 + x_2 + x_3 + \dots + x_n)^{n-1}}. \end{aligned}$$

Solution:

We have

$$\begin{aligned} \frac{x_1^3}{x_2 x_3 \dots x_n (x_2 + x_3 + \dots + x_n)^3} &= \frac{\frac{x_1^4}{x_1 x_2 x_3 \dots x_n (x_2 + x_3 + \dots + x_n)^2}}{x_2 + x_3 + \dots + x_n} \\ \frac{x_2^3}{x_1 x_3 \dots x_n (x_1 + x_3 + \dots + x_n)^3} &= \frac{\frac{x_2^4}{x_1 x_2 x_3 \dots x_n (x_1 + x_3 + \dots + x_n)^2}}{x_1 + x_3 + \dots + x_n} \\ &\dots \\ \frac{x_n^3}{x_1 x_2 \dots x_{n-1} (x_1 + x_2 + \dots + x_{n-1})^3} &= \frac{\frac{x_n^4}{x_1 x_2 \dots x_{n-1} x_n (x_1 + x_2 + \dots + x_{n-1})^2}}{x_1 + x_2 + \dots + x_{n-1}} \end{aligned}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\begin{aligned} &\frac{x_1^3}{x_2 x_3 \dots x_n (x_2 + x_3 + \dots + x_n)^3} + \frac{x_2^3}{x_1 x_3 \dots x_n (x_1 + x_3 + \dots + x_n)^3} + \dots \\ &+ \frac{x_n^3}{x_1 x_2 \dots x_{n-1} (x_1 + x_2 + \dots + x_{n-1})^3} \\ &\geq \frac{\left(\frac{x_1^2}{x_2 + x_3 + \dots + x_n} + \frac{x_2^2}{x_1 + x_3 + \dots + x_n} + \dots + \frac{x_n^2}{x_1 + x_2 + \dots + x_{n-1}} \right)^2}{(n-1)x_1 x_2 x_3 \dots x_n (x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n)} \end{aligned}$$

On the other hand

$$\begin{aligned} &\frac{x_1^2}{x_2 + x_3 + \dots + x_n} + \frac{x_2^2}{x_1 + x_3 + \dots + x_n} + \dots + \frac{x_n^2}{x_1 + x_2 + \dots + x_{n-1}} \\ &\geq \frac{(x_1 + x_2 + \dots + x_n)^2}{(n-1)(x_1 + x_2 + \dots + x_n)} = \frac{x_1 + x_2 + \dots + x_n}{n-1} \end{aligned}$$

Applying the inequality *AM-GM*, we have

$$x_1 x_2 \dots x_n \leq \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^n$$

Deduce

$$\begin{aligned} &\frac{\left(\frac{x_1^2}{x_2 + x_3 + \dots + x_n} + \frac{x_2^2}{x_1 + x_3 + \dots + x_n} + \dots + \frac{x_n^2}{x_1 + x_2 + \dots + x_{n-1}} \right)^2}{(n-1)x_1 x_2 x_3 \dots x_n (x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n)} \\ &\geq \frac{x_1 + x_2 + x_3 + \dots + x_n}{(n-1)^3 \left(\frac{x_1 + x_2 + x_3 + \dots + x_{n-1} + x_n}{n} \right)^n} \\ &= \frac{n^n}{(n-1)^3 (x_1 + x_2 + x_3 + \dots + x_n)^{n-1}} \end{aligned}$$

As such

$$\frac{x_1^3}{x_2 x_3 \dots x_n (x_2 + x_3 + \dots + x_n)^3} + \frac{x_2^3}{x_1 x_3 \dots x_n (x_1 + x_3 + \dots + x_n)^3} + \dots$$

$$+ \frac{x_n^3}{x_1 x_2 \dots x_{n-1} (x_1 + x_2 + \dots + x_{n-1})^3} \geq \frac{n^n}{(n-1)^3 (x_1 + x_2 + x_3 + \dots + x_n)^{n-1}}.$$

Problem 3: Let x_1, x_2, \dots, x_n be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_4)^4} + \dots + \frac{x_n^4}{(x_2 + x_1)^4} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{16(x_1^2 + x_2^2 + \dots + x_n^2)}.$$

Solution:

We have

$$\frac{x_1^4}{(x_2 + x_3)^4} = \frac{x_1^4}{(x_2 + x_3)^2} \cdot \frac{1}{(x_2 + x_3)^2}$$

$$\frac{x_2^4}{(x_3 + x_4)^4} = \frac{x_2^4}{(x_3 + x_4)^2} \cdot \frac{1}{(x_3 + x_4)^2}$$

.....

$$\frac{x_n^4}{(x_2 + x_1)^4} = \frac{x_n^4}{(x_2 + x_1)^2} \cdot \frac{1}{(x_2 + x_1)^2}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_4)^4} + \dots + \frac{x_n^4}{(x_2 + x_1)^4} \geq \frac{\left(\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_4} + \dots + \frac{x_n^2}{x_2 + x_1} \right)^2}{(x_2 + x_3)^2 + (x_3 + x_4)^2 + \dots + (x_2 + x_1)^2}$$

On the other hand

$$\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_4} + \dots + \frac{x_n^2}{x_2 + x_1} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{2(x_1 + x_2 + \dots + x_n)} = \frac{x_1 + x_2 + \dots + x_n}{2}$$

$$(x_2 + x_3)^2 + (x_3 + x_4)^2 + \dots + (x_2 + x_1)^2 \leq 4(x_1^2 + x_2^2 + \dots + x_n^2)$$

Deduce

$$\frac{\left(\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_4} + \dots + \frac{x_n^2}{x_2 + x_1} \right)^2}{(x_2 + x_3)^2 + (x_3 + x_4)^2 + \dots + (x_2 + x_1)^2} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{4(x_1^2 + x_2^2 + \dots + x_n^2)}$$

As such

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_4)^4} + \dots + \frac{x_n^4}{(x_2 + x_1)^4} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{16(x_1^2 + x_2^2 + \dots + x_n^2)}$$

Problem 4: Let x_1, x_2, x_3 be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_1)^4} + \frac{x_3^4}{(x_1 + x_2)^4} \geq \frac{81x_1x_2x_3}{16(x_1 + x_2 + x_3)^3}.$$

Solution:

We have inequality

$$a^4 + b^4 + c^4 \geq abc(a + b + c) \quad \forall a, b, c > 0$$

Thus

$$\begin{aligned} & \frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_1)^4} + \frac{x_3^4}{(x_1 + x_2)^4} \\ & \geq \frac{x_1x_2x_3}{(x_2 + x_3)(x_3 + x_1)(x_1 + x_2)} \left(\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2} \right) \end{aligned}$$

On the other hand

$$\begin{aligned} & \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2} = \frac{x_1^2}{x_1(x_2 + x_3)} + \frac{x_2^2}{x_2(x_3 + x_1)} + \frac{x_3^2}{x_3(x_1 + x_2)} \\ & \geq \frac{(x_1 + x_2 + x_3)^2}{2(x_1x_2 + x_2x_3 + x_3x_1)} \geq \frac{3(x_1x_2 + x_2x_3 + x_3x_1)}{2(x_1x_2 + x_2x_3 + x_3x_1)} = \frac{3}{2} \\ & \frac{x_1x_2x_3}{(x_2 + x_3)(x_3 + x_1)(x_1 + x_2)} \geq \frac{x_1x_2x_3}{\left(\frac{x_2 + x_3 + x_3 + x_1 + x_1 + x_2}{3} \right)^3} = \frac{27x_1x_2x_3}{8(x_1 + x_2 + x_3)^3} \end{aligned}$$

As such

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_1)^4} + \frac{x_3^4}{(x_1 + x_2)^4} \geq \frac{81x_1x_2x_3}{16(x_1 + x_2 + x_3)^3}$$

Problem 5: Let x_1, x_2, \dots, x_n be positive real numbers. Prove that

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_1)^4} + \frac{x_3^4}{(x_1 + x_2)^4} \geq \frac{(x_1 + x_2 + x_3)^2}{16(x_1^2 + x_2^2 + x_3^2)}.$$

Solution:

We have

$$\frac{x_1^4}{(x_2 + x_3)^4} = \frac{\frac{x_1^4}{(x_2 + x_3)^2}}{(x_2 + x_3)^2}$$

$$\frac{x_2^4}{(x_3 + x_1)^4} = \frac{\frac{x_2^4}{(x_3 + x_1)^2}}{(x_3 + x_1)^2}$$

$$\frac{x_3^4}{(x_1 + x_2)^4} = \frac{\frac{x_3^4}{(x_1 + x_2)^2}}{(x_1 + x_2)^2}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_1)^4} + \frac{x_3^4}{(x_1 + x_2)^4} \geq \frac{\left(\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_1} + \frac{x_3^2}{x_1 + x_2} \right)^2}{(x_2 + x_3)^2 + (x_3 + x_1)^2 + (x_1 + x_2)^2}$$

On the other hand

$$\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_1} + \frac{x_3^2}{x_1 + x_2} \geq \frac{(x_1 + x_2 + x_3)^2}{2(x_1 + x_2 + x_3)} = \frac{x_1 + x_2 + x_3}{2}$$

$$(x_2 + x_3)^2 + (x_3 + x_1)^2 + (x_1 + x_2)^2 \leq 4(x_1^2 + x_2^2 + x_3^2)$$

Deduce

$$\frac{\left(\frac{x_1^2}{x_2 + x_3} + \frac{x_2^2}{x_3 + x_1} + \frac{x_3^2}{x_1 + x_2} \right)^2}{(x_2 + x_3)^2 + (x_3 + x_1)^2 + (x_1 + x_2)^2} \geq \frac{(x_1 + x_2 + x_3)^2}{4(x_1^2 + x_2^2 + x_3^2)}$$

As such

$$\frac{x_1^4}{(x_2 + x_3)^4} + \frac{x_2^4}{(x_3 + x_1)^4} + \frac{x_3^4}{(x_1 + x_2)^4} \geq \frac{(x_1 + x_2 + x_3)^2}{16(x_1^2 + x_2^2 + x_3^2)}$$

Problem 6: Let x_1, x_2, x_3 be positive real numbers. Prove that

$$81x_1x_2x_3(x_1^2 + x_2^2 + x_3^2) \leq (x_1 + x_2 + x_3)^5.$$

Problem 7: Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{bcd(b+c+d)^3} + \frac{b}{cda(c+d+a)^3} + \frac{c}{dab(d+a+b)^3} + \frac{d}{abc(a+b+c)^3}$$

$$\geq \frac{4}{27} \left(\frac{4}{a+b+c+d} \right)^5$$

Solution:

We have

$$\frac{a}{bcd(b+c+d)^3} = \frac{a^2}{abcd(b+c+d)^3}, \quad \frac{b}{cda(c+d+a)^3} = \frac{b^2}{abcd(c+d+a)^3}$$

$$\frac{c}{dab(d+a+b)^3} = \frac{c^2}{abcd(d+a+b)^3}, \quad \frac{d}{abc(a+b+c)^3} = \frac{d^2}{abcd(a+b+c)^3}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{a}{bcd(b+c+d)^3} + \frac{b}{cda(c+d+a)^3} + \frac{c}{dab(d+a+b)^3} + \frac{d}{abc(a+b+c)^3}$$

$$\geq \frac{\left(\frac{a}{b+c+d} + \frac{b}{c+d+a} + \frac{c}{d+a+b} + \frac{d}{a+b+c} \right)^2}{3abcd(a+b+c+d)}$$

On the other hand

$$\frac{a}{b+c+d} + \frac{b}{c+d+a} + \frac{c}{d+a+b} + \frac{d}{a+b+c} \geq \frac{4}{3}$$

Applying the inequality

$$abcd \leq \left(\frac{a+b+c+d}{4} \right)^4$$

As such

$$\frac{\left(\frac{a}{b+c+d} + \frac{b}{c+d+a} + \frac{c}{d+a+b} + \frac{d}{a+b+c} \right)^2}{abcd(a+b+c+d)} \geq \frac{\left(\frac{4}{3} \right)^2}{\left(\frac{a+b+c+d}{4} \right)^4 (a+b+c+d)}$$

Thus

$$\frac{a}{bcd(b+c+d)^3} + \frac{b}{cda(c+d+a)^3} + \frac{c}{dab(d+a+b)^3} + \frac{d}{abc(a+b+c)^3}$$

$$\geq \frac{4}{27} \left(\frac{4}{a+b+c+d} \right)^5$$

Problem 8: Let a, b, c, d be positive real numbers. Prove that

$$\frac{1}{bcd(b+2c+3d)^3} + \frac{1}{cda(c+2d+3a)^3} + \frac{1}{dab(d+2a+3b)^3} + \frac{1}{abc(a+2b+3c)^3}$$

$$\geq \frac{1}{54} \left(\frac{4}{a+b+c+d} \right)^6$$

Solution:

We have

$$\frac{1}{bcd(b+2c+3d)^3} = \frac{\frac{a^2}{(b+2c+3d)^2}}{abcd(ab+2ac+3ad)}, \quad \frac{1}{cda(c+2d+3a)^3} = \frac{\frac{b^2}{(c+2d+3a)^2}}{abcd(bc+2bd+3ab)}$$

$$\frac{1}{dab(d+2a+3b)^3} = \frac{\frac{c^2}{(d+2a+3b)^2}}{abcd(cd+2ca+3bc)}, \quad \frac{1}{abc(a+2b+3c)^3} = \frac{\frac{d^2}{(a+2b+3c)^2}}{abcd(ad+2bd+3cd)}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\begin{aligned} & \frac{1}{bcd(b+2c+3d)^3} + \frac{1}{cda(c+2d+3a)^3} + \frac{1}{dab(d+2a+3b)^3} + \frac{1}{abc(a+2b+3c)^3} \\ & \geq \frac{\left(\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \right)^2}{4abcd(ab+ac+ad+bc+bd+cd)} \end{aligned}$$

On the other hand

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

Applying the inequality

$$\begin{aligned} abcd & \leq \left(\frac{a+b+c+d}{4} \right)^4 \\ ab+ac+ad+bc+bd+cd & \leq \frac{3}{8}(a+b+c+d)^2 \end{aligned}$$

As such

$$\begin{aligned} & \frac{\left(\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \right)^2}{abcd(ab+ac+ad+bc+bd+cd)} \\ & \geq \frac{\left(\frac{2}{3} \right)^2}{\left(\frac{a+b+c+d}{4} \right)^4 \frac{3}{8}(a+b+c+d)^2} \end{aligned}$$

Thus

$$\begin{aligned} & \frac{1}{bcd(b+2c+3d)^3} + \frac{1}{cda(c+2d+3a)^3} + \frac{1}{dab(d+2a+3b)^3} + \frac{1}{abc(a+2b+3c)^3} \\ & \geq \frac{1}{54} \left(\frac{4}{a+b+c+d} \right)^6 \end{aligned}$$

Problem 9: Let a, b, c, d be positive real numbers. Prove that

$$\begin{aligned} & \frac{a}{bcd(b+c)^3} + \frac{b}{cda(c+d)^3} + \frac{c}{dab(d+a)^3} + \frac{d}{abc(a+b)^3} \\ & \geq \frac{1}{2} \left(\frac{4}{a+b+c+d} \right)^5 \end{aligned}$$

Solution:

We have

$$\frac{a}{bcd(b+c)^3} = \frac{a^2}{abcd(b+c)^2}, \quad \frac{b}{cda(c+d)^3} = \frac{b^2}{abcd(c+d)^2}$$

$$\frac{c}{dab(d+a)^3} = \frac{c^2}{abcd(d+a)^2}, \quad \frac{d}{abc(a+b)^3} = \frac{d^2}{abcd(a+b)^2}$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{a}{bcd(b+c)^3} + \frac{b}{cda(c+d)^3} + \frac{c}{dab(d+a)^3} + \frac{d}{abc(a+b)^3}$$

$$\geq \frac{\left(\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b}\right)^2}{2abcd(a+b+c+d)}$$

On the other hand

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2$$

Applying the inequality

$$abcd \leq \left(\frac{a+b+c+d}{4}\right)^4$$

As such

$$\frac{\left(\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b}\right)^2}{abcd(a+b+c+d)} \geq \frac{4}{\left(\frac{a+b+c+d}{4}\right)^4 (a+b+c+d)}$$

Thus

$$\frac{a}{bcd(b+c)^3} + \frac{b}{cda(c+d)^3} + \frac{c}{dab(d+a)^3} + \frac{d}{abc(a+b)^3}$$

$$\geq \frac{1}{2} \left(\frac{4}{a+b+c+d}\right)^5$$

===== THE END =====