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*Problem 1:* Let $x_1,\ x_2, x_3$ be non-negative real numbers and satisfy $x_1^2+x_2^2+x_3^2=3$.
Prove that

$$\frac{x_1}{x_1^4+2x_2^2+9}+\frac{x_2}{x_2^4+2x_3^2+9}+\frac{x_3}{x_3^4+2x_1^2+9}\le\frac{1}{4}.$$

*Solution:*

We have

$$x_1^4+2x_2^2+9=x_1^4+1+2\left(x_2^2+1\right)+6\ge 2x_1^2+4x_2+6\ge 4x_1+4x_2+4$$
$$x_2^4+2x_3^2+9\ge x_2^4+1+2\left(x_3^2+1\right)+6\ge 2x_2^2+4x_3+6\ge 4x_2+4x_3+4$$
$$x_3^4+2x_1^2+9\ge x_3^4+1+2\left(x_1^2+1\right)+6\ge 2x_3^2+4x_1+6\ge 4x_3+4x_1+4$$

Deduce

$$\frac{x_1}{x_1^4+2x_2^2+9}+\frac{x_2}{x_2^4+2x_3^2+9}+\frac{x_3}{x_3^4+2x_1^2+9}\le\frac{1}{4}\left(\frac{x_1}{x_1+x_2+1}+\frac{x_2}{x_2+x_3+1}+\frac{x_3}{x_3+x_1+1}\right)$$

We need prove

$$\frac{x_1}{x_1+x_2+1}+\frac{x_2}{x_2+x_3+1}+\frac{x_3}{x_3+x_1+1}\le 1$$

We have

$$\frac{x_1}{x_1+x_2+1}+\frac{x_2}{x_2+x_3+1}+\frac{x_3}{x_3+x_1+1}\le 1$$
$$\Leftrightarrow\frac{x_2+1}{x_1+x_2+1}+\frac{x_3+1}{x_2+x_3+1}+\frac{x_1+1}{x_3+x_1+1}\ge 2$$

By the *Cauchy – Schwarz* inequality, we obtain

$$\frac{x_2+1}{x_1+x_2+1}+\frac{x_3+1}{x_2+x_3+1}+\frac{x_1+1}{x_3+x_1+1}$$
$$\ge\frac{\left(x_1+x_2+x_3+3\right)^2}{\left(x_2+1\right)\left(x_1+x_2+1\right)+\left(x_3+1\right)\left(x_2+x_3+1\right)+\left(x_1+1\right)\left(x_3+x_1+1\right)}$$

On the other hand

$$2\left[\left(x_2+1\right)\left(x_1+x_2+1\right)+\left(x_3+1\right)\left(x_2+x_3+1\right)+\left(x_1+1\right)\left(x_3+x_1+1\right)\right]$$
$$=2x_1^2+2x_2^2+2x_3^2+2x_1x_2+2x_2x_3+2x_3x_1+6x_1+6x_2+6x_3+6$$
$$=\left(x_1+x_2+x_3+3\right)^2$$

As such

$$\frac{x_2+1}{x_1+x_2+1}+\frac{x_3+1}{x_2+x_3+1}+\frac{x_1+1}{x_3+x_1+1}\ge 2$$

Thus

$$\frac{x_1}{x_1^4+2x_2^2+9}+\frac{x_2}{x_2^4+2x_3^2+9}+\frac{x_3}{x_3^4+2x_1^2+9}\le\frac{1}{4}$$

*Problem 2:* Let $x_1, x_2, x_3$ be positive real numbers. Prove that

$$\frac{1}{2x_1^2+x_2^2+x_3^2+4}+\frac{1}{x_1^2+2x_2^2+x_3^2+4}+\frac{1}{x_1^2+x_2^2+2x_3^2+4}\le\frac{1}{8}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right).$$

*Solution:*

We have

$$2x_1^2+x_2^2+x_3^2+4=2\left(x_1^2+1+\frac{x_2^2+x_3^2+2}{2}\right)\ge 2\left(2x_1+x_2+x_3\right)$$

$$x_1^2+2x_2^2+x_3^2+4=2\left(x_2^2+1+\frac{x_3^2+x_1^2+2}{2}\right)\ge 2\left(x_1+2x_2+x_3\right)$$

$$x_1^2+x_2^2+2x_3^2+4=2\left(x_3^2+1+\frac{x_1^2+x_2^2+2}{2}\right)\ge 2\left(x_1+x_2+2x_3\right)$$

As such

$$\frac{1}{2x_1^2+x_2^2+x_3^2+4}+\frac{1}{x_1^2+2x_2^2+x_3^2+4}+\frac{1}{x_1^2+x_2^2+2x_3^2+4}$$

$$\le\frac{1}{2}\left(\frac{1}{2x_1+x_2+x_3}+\frac{1}{x_1+2x_2+x_3}+\frac{1}{x_1+x_2+2x_3}\right)$$

By the $AM-GM$ inequality, we obtain

$$\frac{1}{2x_1+x_2+x_3}=\frac{1}{\left(x_1+x_2\right)+\left(x_1+x_3\right)}\le\frac{1}{4}\left(\frac{1}{x_1+x_2}+\frac{1}{x_1+x_3}\right)$$

$$\le\frac{1}{16}\left(\frac{2}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$$

Similarly

$$\frac{1}{x_1+2x_2+x_3}\le\frac{1}{16}\left(\frac{1}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}\right),\ \frac{1}{x_1+x_2+2x_3}\le\frac{1}{16}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{2}{x_3}\right)$$

As such

$$\frac{1}{2x_1+x_2+x_3}+\frac{1}{x_1+2x_2+x_3}+\frac{1}{x_1+x_2+2x_3}\le\frac{1}{4}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$$

Thus

$$\frac{1}{2x_1^2+x_2^2+x_3^2+4}+\frac{1}{x_1^2+2x_2^2+x_3^2+4}+\frac{1}{x_1^2+x_2^2+2x_3^2+4}\le\frac{1}{8}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$$

*Problem 3:* Let $x_1, x_2, x_3$ be positive real numbers and satisfy $\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=3$.

Prove that

$$\frac{1}{2x_1^2+x_2^2+x_3^2+4}+\frac{1}{x_1^2+2x_2^2+x_3^2+4}+\frac{1}{x_1^2+x_2^2+2x_3^2+4}\le\frac{3}{8}.$$

*Solution:*

We have

$$2x_1^2+x_2^2+x_3^2+4=2\left(x_1^2+1+\frac{x_2^2+x_3^2+2}{2}\right)\geq 2\left(2x_1+x_2+x_3\right)$$

$$x_1^2+2x_2^2+x_3^2+4=2\left(x_2^2+1+\frac{x_3^2+x_1^2+2}{2}\right)\geq 2\left(x_1+2x_2+x_3\right)$$

$$x_1^2+x_2^2+2x_3^2+4=2\left(x_3^2+1+\frac{x_1^2+x_2^2+2}{2}\right)\geq 2\left(x_1+x_2+2x_3\right)$$

As such

$$\frac{1}{2x_1^2+x_2^2+x_3^2+4}+\frac{1}{x_1^2+2x_2^2+x_3^2+4}+\frac{1}{x_1^2+x_2^2+2x_3^2+4}$$

$$\leq\frac{1}{2}\left(\frac{1}{2x_1+x_2+x_3}+\frac{1}{x_1+2x_2+x_3}+\frac{1}{x_1+x_2+2x_3}\right)$$

By the $AM-GM$ inequality, we obtain

$$\frac{1}{2x_1+x_2+x_3}=\frac{1}{\left(x_1+x_2\right)+\left(x_1+x_3\right)}\leq\frac{1}{4}\left(\frac{1}{x_1+x_2}+\frac{1}{x_1+x_3}\right)$$

$$\leq\frac{1}{16}\left(\frac{2}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$$

Similarly

$$\frac{1}{x_1+2x_2+x_3}\leq\frac{1}{16}\left(\frac{1}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}\right),\ \frac{1}{x_1+x_2+2x_3}\leq\frac{1}{16}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{2}{x_3}\right)$$

As such

$$\frac{1}{2x_1+x_2+x_3}+\frac{1}{x_1+2x_2+x_3}+\frac{1}{x_1+x_2+2x_3}\leq\frac{1}{4}\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$$

Thus

$$\frac{1}{2x_1^2+x_2^2+x_3^2+4}+\frac{1}{x_1^2+2x_2^2+x_3^2+4}+\frac{1}{x_1^2+x_2^2+2x_3^2+4}\leq\frac{3}{8}$$

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