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Problem 1: Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc\left(b^2+2c^2\right)}+\frac{b^3}{ca\left(c^2+2a^2\right)}+\frac{c^3}{ab\left(a^2+2b^2\right)}\geq\frac{3}{a+b+c}.$$

Solution:

We have

$$\frac{a^3}{bc\left(b^2+2c^2\right)}=\frac{a^4}{abc\left(b^2+2c^2\right)}$$

$$\frac{b^3}{ca\left(c^2+2a^2\right)}=\frac{b^4}{abc\left(c^2+2a^2\right)}$$

$$\frac{c^3}{ab\left(a^2+2b^2\right)}=\frac{c^4}{abc\left(a^2+2b^2\right)}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{a^3}{bc\left(b^2+2c^2\right)}+\frac{b^3}{ca\left(c^2+2a^2\right)}+\frac{c^3}{ab\left(a^2+2b^2\right)}$$

$$\geq\frac{\left(a^2+b^2+c^2\right)^2}{3abc\left(a^2+b^2+c^2\right)}=\frac{a^2+b^2+c^2}{3abc}$$

Applying inequalities

$$3\left(a^2+b^2+c^2\right)\geq(a+b+c)^2$$

$$abc\leq\left(\frac{a+b+c}{3}\right)^3$$

We have

$$\frac{a^2+b^2+c^2}{3abc}\geq\frac{3}{a+b+c}$$

As such

$$\frac{a^3}{bc\left(b^2+2c^2\right)}+\frac{b^3}{ca\left(c^2+2a^2\right)}+\frac{c^3}{ab\left(a^2+2b^2\right)}\geq\frac{3}{a+b+c}$$

Problem 2: Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc(a+b)^2(a+c)}+\frac{b^3}{ca(b+c)^2(a+b)}+\frac{c^3}{ab(c+a)^2(b+c)}\geq\frac{27}{8(a+b+c)^2}.$$

Solution:

We have

$$\frac{a^3}{bc(a+b)^2(a+c)} = \frac{a^4}{abc(a+b)^2(a+c)}$$
$$\frac{b^3}{ca(b+c)^2(a+b)} = \frac{b^4}{abc(b+c)^2(a+b)}$$
$$\frac{c^3}{ab(c+a)^2(b+c)} = \frac{c^4}{abc(c+a)^2(b+c)}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{a^3}{bc(a+b)^2(a+c)} + \frac{b^3}{ca(b+c)^2(a+b)} + \frac{c^3}{ab(c+a)^2(b+c)}$$
$$\geq \frac{\left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\right)^2}{2abc(a+b+c)}$$

On the other hand

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \geq \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2}$$

As such

$$\frac{\left(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\right)^2}{2abc(a+b+c)} \geq \frac{(a+b+c)^2}{8abc(a+b+c)} = \frac{a+b+c}{8abc}$$

Applying inequality

$$abc \leq \left(\frac{a+b+c}{3}\right)^3$$

We have

$$\frac{a+b+c}{8abc} \geq \frac{27}{8(a+b+c)^2}$$

Thus

$$\frac{a^3}{bc(a+b)^2(a+c)} + \frac{b^3}{ca(b+c)^2(a+b)} + \frac{c^3}{ab(c+a)^2(b+c)} \geq \frac{27}{8(a+b+c)^2}$$

Problem 3: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3bc(a+b)(a+c)} + \frac{1}{ab^3c(b+c)(a+b)} + \frac{1}{abc^3(c+a)(b+c)} \geq \frac{3}{4}\left(\frac{3}{a+b+c}\right)^7.$$

Solution:

We have

$$\frac{1}{a^3bc(a+b)(a+c)}+\frac{1}{ab^3c(b+c)(a+b)}+\frac{1}{abc^3(c+a)(b+c)}$$
$$=\frac{\frac{1}{a^2}}{abc(a+b)(a+c)}+\frac{\frac{1}{b^2}}{abc(b+c)(a+b)}+\frac{\frac{1}{c^2}}{abc(c+a)(b+c)}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{1}{a^3bc(a+b)(a+c)}+\frac{1}{ab^3c(b+c)(a+b)}+\frac{1}{abc^3(c+a)(b+c)}$$
$$\geq\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{abc\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}=\frac{(ab+bc+ca)^2}{(abc)^3\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}$$

Applying inequalities

$$(abc)^3\leq\left(\frac{a+b+c}{3}\right)^3$$

$$a^2+b^2+c^2+3(ab+bc+ca)\leq\frac{4(a+b+c)^2}{3}$$

$$ab+bc+ca\geq\frac{9}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

We have

$$\frac{(ab+bc+ca)^2}{(abc)^3\left(a^2+b^2+c^2+3(ab+bc+ca)\right)}$$
$$\geq\frac{243}{4(abc)^3(a+b+c)^2\left(\frac{a+b+c}{abc}\right)^2}\geq\frac{3}{4}\left(\frac{3}{a+b+c}\right)^7$$

As such

$$\frac{1}{a^3bc(a+b)(a+c)}+\frac{1}{ab^3c(b+c)(a+b)}+\frac{1}{abc^3(c+a)(b+c)}\geq\frac{3}{4}\left(\frac{3}{a+b+c}\right)^7$$

Problem 4: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3bc(b+2c)}+\frac{1}{ab^3c(c+2a)}+\frac{1}{abc^3(a+2b)}\geq\left(\frac{3}{a+b+c}\right)^6.$$

Solution:

We have

$$\frac{1}{a^3bc(b+2c)}+\frac{1}{ab^3c(c+2a)}+\frac{1}{abc^3(a+2b)}$$

$$=\frac{\frac{1}{a^2}}{abc(b+2c)}+\frac{\frac{1}{b^2}}{abc(c+2a)}+\frac{\frac{1}{c^2}}{abc(a+2b)}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{1}{a^3bc(b+2c)}+\frac{1}{ab^3c(c+2a)}+\frac{1}{abc^3(a+2b)}$$

$$\geq \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{3abc(a+b+c)}=\frac{(ab+bc+ca)^2}{3(abc)^3(a+b+c)}$$

Applying inequalities

$$ab+bc+ca\geq \frac{9}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

$$(abc)^3\leq \left(\frac{a+b+c}{3}\right)^3$$

We have

$$\frac{(ab+bc+ca)^2}{3(abc)^3(a+b+c)}$$

$$\geq \frac{81}{3(abc)^3(a+b+c)\left(\frac{a+b+c}{abc}\right)^2}\geq \left(\frac{3}{a+b+c}\right)^6$$

As such

$$\frac{1}{a^3bc(b+2c)}+\frac{1}{ab^3c(c+2a)}+\frac{1}{abc^3(a+2b)}\geq \left(\frac{3}{a+b+c}\right)^6$$

Problem 5: Let a, b, c be positive real numbers. Prove that

$$\frac{a^2+ab+b^2}{a^2bc}+\frac{b^2+bc+c^2}{ab^2c}+\frac{c^2+ca+a^2}{abc^2}\geq \frac{3^6abc}{(a+b+c)^3(a^2+b^2+c^2)}.$$

Solution:

By the Cauchy – Schwarz inequality, we obtain

$$\frac{a^2+ab+b^2}{a^2bc}+\frac{b^2+bc+c^2}{ab^2c}+\frac{c^2+ca+a^2}{abc^2}$$
$$\geq \frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{abc\left(a^3+b^3+c^3+ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)\right)}$$

On the other hand

$$a^3+b^3+c^3+ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)=\left(a+b+c\right)\left(a^2+b^2+c^2\right)$$

Deduce

$$\frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{abc\left(a^3+b^3+c^3+ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)\right)}=\frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{abc\left(a+b+c\right)\left(a^2+b^2+c^2\right)}$$

Applying inequalities

$$a^2+b^2+c^2\geq ab+bc+ca$$
$$ab+bc+ca\geq \frac{9}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

We obtain

$$\frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{abc\left(a^3+b^3+c^3+ab\left(a+b\right)+bc\left(b+c\right)+ca\left(c+a\right)\right)}\geq \frac{9\left(ab+bc+ca\right)^2}{abc\left(a+b+c\right)\left(a^2+b^2+c^2\right)}$$
$$\geq \frac{9.81}{abc\left(a+b+c\right)\left(a^2+b^2+c^2\right)\left(\frac{a+b+c}{abc}\right)^2}=\frac{3^6abc}{\left(a+b+c\right)^3\left(a^2+b^2+c^2\right)}$$

As such

$$\frac{a^2+ab+b^2}{a^2bc}+\frac{b^2+bc+c^2}{ab^2c}+\frac{c^2+ca+a^2}{abc^2}\geq \frac{3^6abc}{\left(a+b+c\right)^3\left(a^2+b^2+c^2\right)}$$

Problem 6: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3\left(a^2+ab+b^2\right)}+\frac{1}{b^3\left(b^2+bc+c^2\right)}+\frac{1}{c^3\left(c^2+ca+a^2\right)}\geq \frac{81}{\left(a+b+c\right)^3\left(a^2+b^2+c^2\right)}.$$

Solution:

We have

$$\frac{1}{a^3\left(a^2+ab+b^2\right)}+\frac{1}{b^3\left(b^2+bc+c^2\right)}+\frac{1}{c^3\left(c^2+ca+a^2\right)}$$
$$=\frac{\frac{1}{a^2}}{a^3+a^2b+ab^2}+\frac{\frac{1}{b^2}}{b^3+b^2c+bc^2}+\frac{\frac{1}{c^2}}{c^3+c^2a+ca^2}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{1}{a^3\left(a^2+ab+b^2\right)}+\frac{1}{b^3\left(b^2+bc+c^2\right)}+\frac{1}{c^3\left(c^2+ca+a^2\right)}$$

$$\geq \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)}$$

$$=\frac{(ab+bc+ca)^2}{(abc)^2\left(a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)\right)}$$

On the other hand

$$a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)=(a+b+c)\left(a^2+b^2+c^2\right)$$

Deduce

$$\frac{(ab+bc+ca)^2}{(abc)^2\left(a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)\right)}=\frac{(ab+bc+ca)^2}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)}$$

Applying inequality

$$ab+bc+ca\geq \frac{9}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

We obtain

$$\frac{(ab+bc+ca)^2}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)}\geq \frac{81}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2}$$

$$=\frac{81}{(a+b+c)^3\left(a^2+b^2+c^2\right)}$$

As such

$$\frac{1}{a^3\left(a^2+ab+b^2\right)}+\frac{1}{b^3\left(b^2+bc+c^2\right)}+\frac{1}{c^3\left(c^2+ca+a^2\right)}\geq \frac{81}{(a+b+c)^3\left(a^2+b^2+c^2\right)}$$

Problem 7: Let a, b, c be positive real numbers. Prove that

$$\frac{a^2+ab+b^2}{a^3b^2c^2}+\frac{b^2+bc+c^2}{a^2b^3}+\frac{c^2+ca+a^2}{a^2b^2c^3}\geq \frac{3^6}{(a+b+c)^3\left(a^2+b^2+c^2\right)}.$$

Solution:

By the Cauchy – Schwarz inequality, we obtain

$$\frac{a^2+ab+b^2}{a^3b^2c^2}+\frac{b^2+bc+c^2}{a^2b^3}+\frac{c^2+ca+a^2}{a^2b^2c^3}$$
$$\geq \frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{(abc)^2\left(a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)\right)}$$

On the other hand

$$a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)=(a+b+c)\left(a^2+b^2+c^2\right)$$

We obtain

$$\frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{(abc)^2\left(a^3+b^3+c^3+ab(a+b)+bc(b+c)+ca(c+a)\right)}=\frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)}$$

Applying inequalities

$$a^2+b^2+c^2\geq ab+bc+ca$$
$$ab+bc+ca\geq \frac{9}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

We obtain

$$\frac{\left(2\left(a^2+b^2+c^2\right)+ab+bc+ca\right)^2}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)}\geq \frac{9(ab+bc+ca)^2}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)}$$
$$\geq \frac{9.81}{(abc)^2(a+b+c)\left(a^2+b^2+c^2\right)\left(\frac{a+b+c}{abc}\right)^2}=\frac{3^6}{(a+b+c)^3\left(a^2+b^2+c^2\right)}$$

As such

$$\frac{a^2+ab+b^2}{a^3b^2c^2}+\frac{b^2+bc+c^2}{a^2b^3}+\frac{c^2+ca+a^2}{a^2b^2c^3}\geq \frac{3^6}{(a+b+c)^3\left(a^2+b^2+c^2\right)}$$

Problem 8: Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^2\left(b^2+2c^2\right)}+\frac{1}{b^2\left(c^2+2a^2\right)}+\frac{1}{c^2\left(a^2+2b^2\right)}\geq\left(\frac{3}{a^2+b^2+c^2}\right)^2$$

Solution:

We have

$$\frac{1}{a^2\left(b^2+2c^2\right)}+\frac{1}{b^2\left(c^2+2a^2\right)}+\frac{1}{c^2\left(a^2+2b^2\right)}$$
$$=\frac{\frac{1}{a^2}}{a^2+2b^2}+\frac{\frac{1}{b^2}}{b^2+2c^2}+\frac{\frac{1}{c^2}}{c^2+2a^2}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{1}{a^2\left(b^2+2c^2\right)}+\frac{1}{b^2\left(c^2+2a^2\right)}+\frac{1}{c^2\left(a^2+2b^2\right)}$$

$$\geq \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{3\left(a^2+b^2+c^2\right)}=\frac{\left(ab+bc+ca\right)^2}{3\left(abc\right)^2\left(a^2+b^2+c^2\right)}$$

Applying inequalities

$$ab+bc+ca \geq \frac{9}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

$$\left(a+b+c\right)^2 \leq 3\left(a^2+b^2+c^2\right)$$

We obtain

$$\frac{\left(ab+bc+ca\right)^2}{3\left(abc\right)^2\left(a^2+b^2+c^2\right)} \geq \frac{27}{\left(abc\right)^2\left(a^2+b^2+c^2\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2}$$

$$=\frac{27}{\left(a+b+c\right)^2\left(a^2+b^2+c^2\right)} \geq \frac{9}{\left(a^2+b^2+c^2\right)^2}$$

As such

$$\frac{1}{a^2\left(b^2+2c^2\right)}+\frac{1}{b^2\left(c^2+2a^2\right)}+\frac{1}{c^2\left(a^2+2b^2\right)} \geq \left(\frac{3}{a^2+b^2+c^2}\right)^2$$

Problem 9: Let a, b, c be positive real numbers and satisfy the condition

$$a+b+c=3.$$

Prove that

$$\frac{a^6}{b^2+2c^2}+\frac{b^6}{c^2+2a^2}+\frac{c^6}{a^2+2b^2} \geq 1.$$

Solution:

We have

$$\frac{a^6}{b^2+2c^2}=\frac{a^8}{a^2\left(b^2+2c^2\right)}$$

$$\frac{b^6}{c^2+2a^2}=\frac{b^8}{b^2\left(c^2+2a^2\right)}$$

$$\frac{c^6}{a^2+2b^2}=\frac{c^8}{c^2\left(a^2+2b^2\right)}$$

By the Cauchy – Schwarz inequality, we obtain

$$\frac{a^6}{b^2+2c^2}+\frac{b^6}{c^2+2a^2}+\frac{1}{a^2+2b^2}\ge\frac{\left(a^4+b^4+c^4\right)^2}{3\left(a^2b^2+b^2c^2+c^2a^2\right)}$$

Applying inequality

$$a^2b^2+b^2c^2+c^2a^2\le a^4+b^4+c^4$$

Deduce

$$\frac{\left(a^4+b^4+c^4\right)^2}{3\left(a^2b^2+b^2c^2+c^2a^2\right)}\ge\frac{a^4+b^4+c^4}{3}$$

As such

$$\frac{a^6}{b^2+2c^2}+\frac{b^6}{c^2+2a^2}+\frac{1}{a^2+2b^2}\ge\frac{a^4+b^4+c^4}{3}$$

On the other hand

$$\frac{a^4+b^4+c^4}{3}\ge\left(\frac{a+b+c}{3}\right)^4=1$$

Thus

$$\frac{a^6}{b^2+2c^2}+\frac{b^6}{c^2+2a^2}+\frac{c^6}{a^2+2b^2}\ge 1$$

Problem 10: Let a, b, c be non-negative real numbers. Prove that

$$3abc\left(ab+bc+ca\right)\ge\left(a+b+c\right)^2\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right).$$

Problem 11: Let a, b, c be non-negative real numbers. Prove that

$$9a^2b^2c^2\ge\left(a+b+c\right)\left(ab+bc+ca\right)\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right).$$

Problem 12: Let a, b, c be non-negative real numbers. Prove that

$$4abc\left(ab+bc+ca\right)\ge\left(a^2+b^2+c^2+3\left(ab+bc+ca\right)\right)\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right).$$

Problem 13: Let a, b, c be non-negative real numbers. Setting

$$M=\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right)$$

$$N=\left(a+b-c\right)^2+\left(b+c-a\right)^2+\left(c+a-b\right)^2$$

$$P=abc$$

$$Q=a^2+b^2+c^2$$

Prove that

$$MN\le PQ.$$

Problem 14: Let a, b, c be non-negative real numbers. Prove that

$$8a^3b^3c^3\ge\left(a^2+b^2\right)\left(b^2+c^2\right)\left(c^2+a^2\right)\left(a+b-c\right)\left(b+c-a\right)\left(c+a-b\right).$$

Problem 15: Let a, b, c be non-negative real numbers. Prove that

$$3abc\left(a^2+b^2+c^2+3(ab+bc+ca)\right)\geq 4(a+b+c)^2(a+b-c)(b+c-a)(c+a-b).$$

Problem 16: Let a, b, c be non-negative real numbers. Prove that

$$3abc(a+b+c)(ab+bc+ca)\geq 3\left(a^3+b^3+c^3\right)(a+b-c)(b+c-a)(c+a-b).$$

Problem 17: Let a, b, c be non-negative real numbers. Prove that

$$3a^2b^2c^2\geq\left(a^3+b^3+c^3-2abc\right)(a+b-c)(b+c-a)(c+a-b).$$