SOLVING TRIG INEQUALITIES - NGHI H NGUYEN 's METHOD

(Authored by Nghi H Nguyen, **Updated Nov. 24, 2020**)

GENERALITIES ON TRIG INEQUALITIES

A trig inequality is an inequality in the form $F(x) \leq 0$, or $F(x) \geq 0$ that contains one or a few trig functions of the arc AM = x. Solving for x means finding all values of the arc x whose trig functions make the inequality true. All these values of x (expressed in radians or degrees) constitute the solution set of the trig inequality.
Solution set of a trig inequality are expressed in the form of intervals, or arc lengths.

Example of trig inequalities:

$\sin x < 1/2$	$\sin x + \cos x \leq \sin 3x$	$\tan x - \cot x > 1$
$\sin x + \sin 2x > \sin 3x$	$\cos^2 x - \sin x \geq -1$	$(2\cos x - 1)(2\cos x + 1) < 0$

Example of intervals, or arc lengths:

Open intervals:	$(0, \pi/3)$	$(\pi/6, 5\pi/6)$	$(\pi/2, 3\pi/2)$
Half closed intervals:	$(-\infty, \pi/6]$	$[\pi/3, +\infty)$	$[0, +\infty)$
Closed intervals:	$[\pi/3, 3\pi/2]$	$[20^\circ, 90^\circ]$	$[45^\circ, 120^\circ]$

THE TRIG UNIT CIRCLE AND BASIC TRIG FUNCTIONS

It is a circle with radius R = 1, and with an origin O. The unit circle defines the main trig functions of the variable arc x that varies and rotates counterclockwise on it.
On the unit circle, the value of the **arc x** is exactly **equal** to the corresponding **angle x**. It is more convenient to use the arc x as the variable, instead of the angle x.
When the variable arc AM = x (in radians or degree) varies on the trig unit circle:

- The horizontal axis OAx defines the function **f(x) = cos x.** When the arc x varies from 0 to 2π, the function f(x) = cos x varies from 1 to -1, then, back to 1.
- The vertical axis OBy defines the function **f(x) = sin x**. When the arc x varies from 0 to 2π, the function f(x) = sin x varies from 0 to 1, then to -1, then back to 0.
- The vertical axis AT defines the function **f(x) = tan x.** When x varies from $-\pi/2$ to $\pi/2$, the function f(x) varies from $-\infty$ to $+\infty$.
- The horizontal axis BU defines the function **f(x) = cot x**. When x varies from 0 to π, the function f(x) = cot x varies from $+\infty$ to $-\infty$.

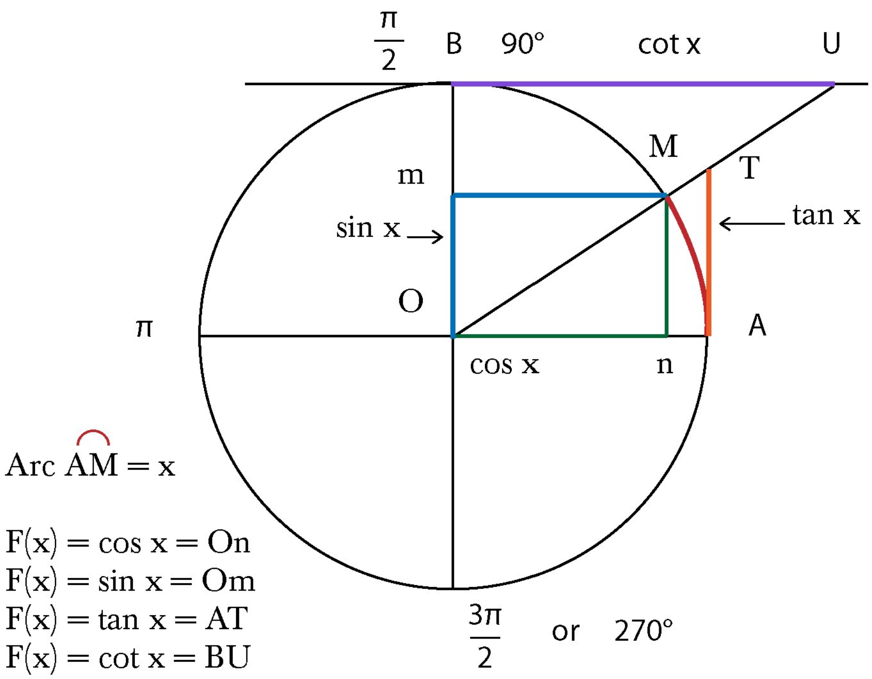


Figure 1

CONCEPT OF SOLVING TRIG INEQUALITIES.

To solve a trig inequality, transform it into one or many basic inequalities. Solving trig inequalities finally results in solving trig basic inequalities.
There are 4 basic trig inequalities:

$\sin x \geq a$ (or $\leq a$)	$\cos x \leq a$ (or $\geq a$)
$\tan x \leq a$ (or $\geq a$)	$\cot x \leq a$ (or $\geq a$)

There are many similar basic trig inequalities that contains one trig function of the arc x. These inequalities can be solved as basic inequalities.

Examples:

$\sin (x + \pi/3) < 1/2$	$\cos (x - 45^{o}) > 2/3$	$2 \sin 2x < 1$
$\cos (2x - 30^{o}) > 0$	$3\tan 2x > 2$	$2\cot (x - \pi/3) > 1$

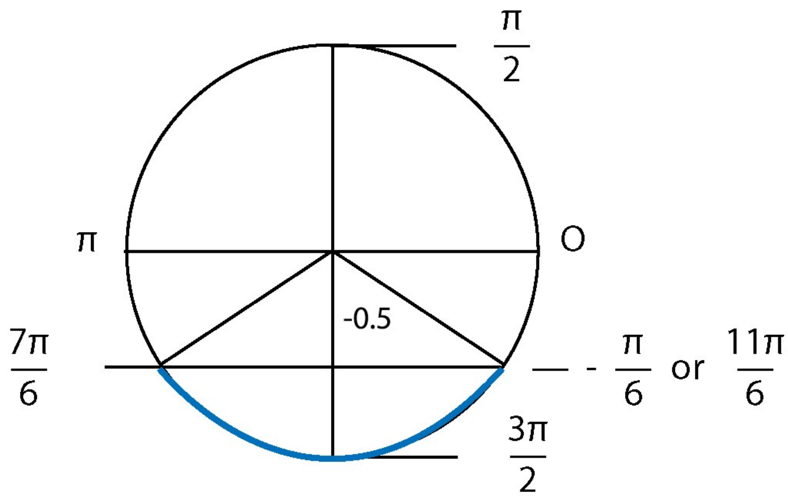
SOLVING TRIG BASIC INEQUALITIES BY THE NUMBER UNIT CIRCLE.

This method finds the solution set of the trig inequality by using the unit circle and the 4 axis. It proceeds exactly like the number line check point method. The unit circle is numbered in radians or in degrees.

Example 1. Solve $\sin x < -1/2$

Solution. Put the inequality in standard form: $F(x) = \sin x + 1/2 < 0$
First step, solve $F(x) = \sin x + 1/2 = 0$ --> $\sin x = -1/2$.
On the sin axis, take $\sin x = -1/2$.
The Table 1:"Values of trig functions of special arcs" and the unit circle give 2 end points at $(-\pi/6)$ and $(7\pi/6)$ that divides the circle into 2 arc lengths
Next, use the check point method to find which arc length is the solution set. Select the check point $(3\pi/2)$. We get: $F(x) = \sin(3\pi/2) + 1/2 = -1 + 1/2 < 0$. It is True. Then the solution set of the trig inequality is the open interval $(7\pi/6, -\pi/6)$ or **$(7\pi/6, 11\pi/6)$**, using co-terminal rule.

Figure 2



Example 2. Solve $\tan x < 3/4$.

<u>Solution</u>. On the tangent axis, get AT = 0.75. Calculator give arc AM1 = 36^o87 and arc AM2 = $180^o + 36^o87 = 216^o87$.
By considering the unit circle, we see that $F(x) = \tan x - 3/4 < 0$ when arc x varies inside the arc length $(-90^o, 36^o87)$.

The answer is: $270^o < x < (360^o + 36^o87 = 396^o87)$
The extended answer is $270^o + k180^o < x < 396^o87 + k180^o$
For k =1, there is another arc length $(90^o, 216^o87)$ that makes the inequality true.

Finally, the solution set of the inequality **F(x) < 0** for $(0, 360^o)$ is 2 open intervals
$(90^o, 216^o87)$ and **$(270^o, 396^o87)$**
Note. We can use the point A(0) as check point. We get: $F(0) = \tan(0) - 3/4 = 0 - 3/4 < 0$. It is true, then, the point A is located on the solution set

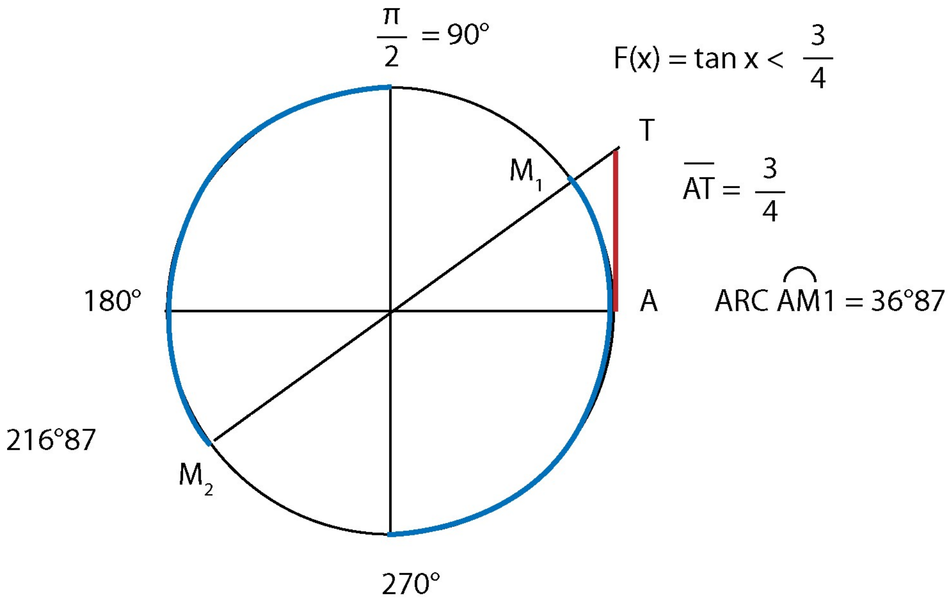


Figure 3

SOLVING COMPLEX TRIG INEQUALITIES BY THE BI-UNIT CIRCLE.

Usually, to solve a complex trig inequality, algebraically, we have to create a sign chart (sign table) that is not easy to handle. This article introduces a new second method that is much visual than the sign chart. It also shows the periodic characteristics of the trig functions.

SOLVING PROCESS.

Usually, to solve a trig inequality in the form $F(x) \geq 0$ (or ≤ 0), we transform it into a product of simple trig functions in the form:

$F(x) = f(x).g(x) \geq 0$ (or ≤ 0) or $F(x) = f(x).g(x).h(x) \geq 0$ (or ≤ 0)

The new method proceeds to find the sign status (+ or -) of the trig functions f(x), g(x), or h(x) when x varies inside the various arc lengths on the unit circle. Next, it colors these various arc lengths, red for positive sign and blue for negative sign.
Then, by superimposing, the resulting sign of F(x), or the combined solution set can be easily seen.
To find the solution set of each simple trig function, it proceeds by the same process as the number line check point method. The difference is instead of a number line, it is a numbered unit circle that can be numbered in radians or in degrees.

Example 3. Solve $\sin x + \sin 2x < - \sin 3x$

<u>Solution</u>. Standard form: $F(x) = \sin x + \sin 2x + \sin 3x < 0$ Common period 2π

First step, using trig "Sum to Product" identity ($\sin a + \sin b$) to transform equation $F(x) = 0$

$F(x) = \sin x + \sin 3x + \sin 2x = 2\sin 2x\cos x + \sin 2x = \sin 2x\,(2\cos x + 1) = 0$
F(x) is in the form $F(x) = f(x).g(x) = 0$. Solve the 2 basic trig equations f(x) and g(x).

1. Solve $f(x) = \sin 2x = 0$ There are 4 end points: $x = 0$; $x = \pi/2$; $x = \pi$; and $x = 3\pi/2$

When x is inside the 4 equal arc lengths:
$(0, \pi/2) \rightarrow f(x) = \sin 2x > 0$. color it red.
$(\pi/2, \pi)$ --> $f(x) = \sin 2x < 0$ (blue)
$(\pi, 3\pi/2)$ --> $f(x) > 0$ (red)
$(3\pi/2, 2\pi)$ --> $f(x) < 0$ (blue)

2. Solve: $g(x) = (2\cos x + 1) = 0 \rightarrow \cos x = -1/2 \rightarrow$ two end points: $x = 2\pi/3$ and $x = 4\pi/3$
$g(x) = 2\cos x + 1 < 0$ when x varies inside the arc length: $(2\pi/3 < x < 4\pi/3)$ (blue)
$g(x) > 0$ the rest of the circle (red). We can use the check point: $x = 0$ --> $g(x) = 3 + 1 > 0$

Figure 4

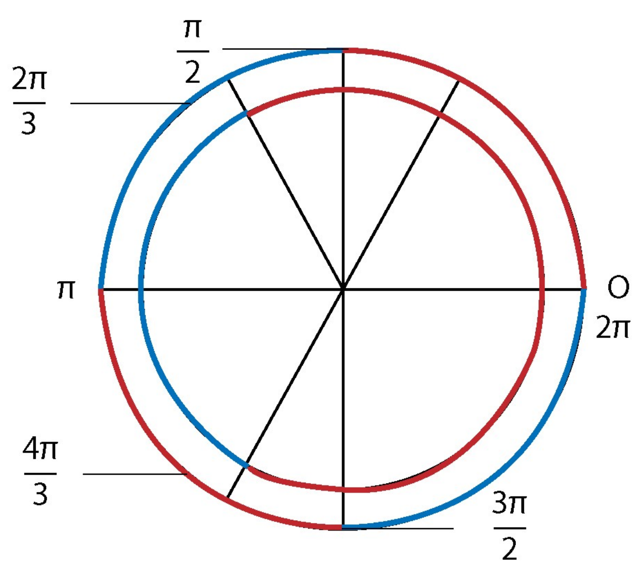


Figure the sign status (red or blue) of f(x) and g(x) on two concentric unit circles.
By superimposing, we see that $F(x) = f(x).g(x) < 0$ when x varies inside the solution set that are the 3 open intervals:

(Л/2, 2Л3) where f(x) < 0 and g(x) > 0 and
(Л, 4Л/3) where f(x) > 0 and g(x) < 0, and
(3Л/2, 2Л) where f(x) < 0 and g(x) > 0

Example 4. Solve sin x + sin 2x + sin 3x < cos x + cos 2x + cos 3x. (0 < x< 2Л)

Solution. Use Sum to Product Identity (sin a + sin b) and (cos a + cos b) to transform:

(sin x + sin 3x) + sin 2x < (cos x + cos 3x) + cos 2x Common period (2Л)
sin 2x(2cos x + 1) < cos 2x (2cos x + 1)
F(x) = f(x).g(x) = (2cos x + 1)(sin 2x – cos 2x) < 0

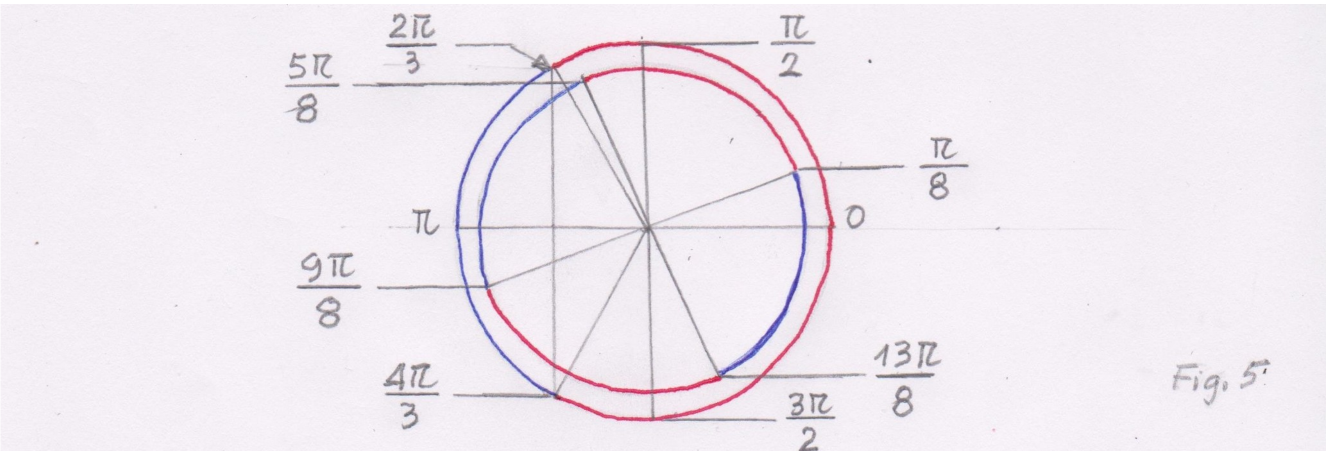
1. Solve f(x) = (2cos x + 1) = 0 --> cos x = - 1/2 --> 2 end points: x = 2Л/3 and x = 4Л/3
Use check point x = Л --> f(x) = 2cos (Л) + 1 = - 2 + 1 < 0
Then, f(x) < 0 inside the interval (2Л/3, 4Л/3) (blue).
The solution set for the rest of the circle --> f(x) > 0 is colored red

2. Solve. g(x) = sin 2x – cos 2x = 0 → Divide by cos 2x (condition cos 2x ≠ 0)
We get: tan 2x - 1 = 0 → tan 2x = 1 -->2 solutions and 2 end points: 2x = Л/4 and 2x = 5Л/4

2x = Л/4 + kЛ --> x = Л/8 + kЛ/2 and
2x = 5Л/4 + kЛ --> x = 5Л/8 + kЛ/2
For k = 0; k = 1, and k = 2 → There are 4 end points at: Л/8, 5Л/8, 9Л/8, and 13Л/8

Use check point x = Л/2. We get g(Л/2) = sin Л/2 – cos Л/2 = 1 – 0 > 0. Then g(x) > 0 inside the interval (Л/8, 5Л/8). Color this arc length red. Color the 3 other arc lengths, following the property of end points.

Figure 5



By superimposing, we see that the solution set are the 3 open intervals:

(5Л/8, 2Л/3) where $f(x) > 0$ and $g(x) < 0$, and
(9Л/8, 4Л/3) where $f(x) < 0$ and $g(x) > 0$
(13Л/8, Л/8 + 2Л), or **(13Л/8, 17Л/8)** where $f(x) > 0$ and $g(x) < 0$

NOTE. Students can use the **triple unit circle** to solve **complex trig inequalities** that are in the form:
F(x) = f(x).g(x).h(x) ≤ 0 (or ≥ 0) where f(x), g(x), and h(x) are basic trig functions, or similar.

Example 5. Solve **sin 6x – sin 4x < sin 2x** Common period (0, 2Л)

Solution. Common period: 2Л. Use these trig identities to transform both sides:
sin 2a = 2sin a.cos a
sin 6a– sin 4a = 2cos 5a.sin a
We get:
2cos 5x.sin x -< 2sin x.cos x
F(x) = 2cos 5x.sin x – 2sin x.cos x < 0
F(x) = 2sin x(cos 5x – cos x) < 0
F(x) = 2sin x(- 2sin 3x.sin 2x) < 0
F(x) = f(x).g(x).h(x) = 4sin 3x.sin 2x.sin x > 0 (side transposing)

Next, find the sign status (+ or -) of f(x), g(x), and h(x) within period (0, 2Л)

1. Solve f(x) = sin 3x = 0.
There are 6 endpoints at: (0), (Л/3). (2Л/3), (Л), (4Л/3), (5Л/3),
There are 6 equal arc lengths. We can use the property of endpoints to color them.

Arc length (0, Л/3) → **f(x) > 0 (red)**
Arc length (Л/3, 2Л/3) → f**(x) < 0 (blue)**
Arc length (2Л/3, Л) → f(x) > 0 (red)
Arc length (Л, 4Л/3) → f(x) < 0 (blue)
Arc length (4Л/3, 5Л/3) → f(x) > 0 (red)
Arc length (5Л/3, 2Л) → f(x) < 0 (blue)

2. Solve g(x) = sin 2x = 0 **→** 2 There are 4 endpoints at (0), (Л/2), (Л), and (3Л/2)
g(x) = sin 2x > 0 (red) when x varies inside the 2 intervals (0, Л/2) and (Л, 3Л,2)
g(x) = sin 2x < 0 (blue) when x varies inside the 2 arc lengths: (Л/2, Л) and (3Л/2, 2Л)

3. Solve h(x) = sin x = 0 **→** There are 2 endpoints at (0) and (Л)
f(x) > 0 hen x is inside the arc length **(0, Л)**. Color it red. The rest of circle: blue

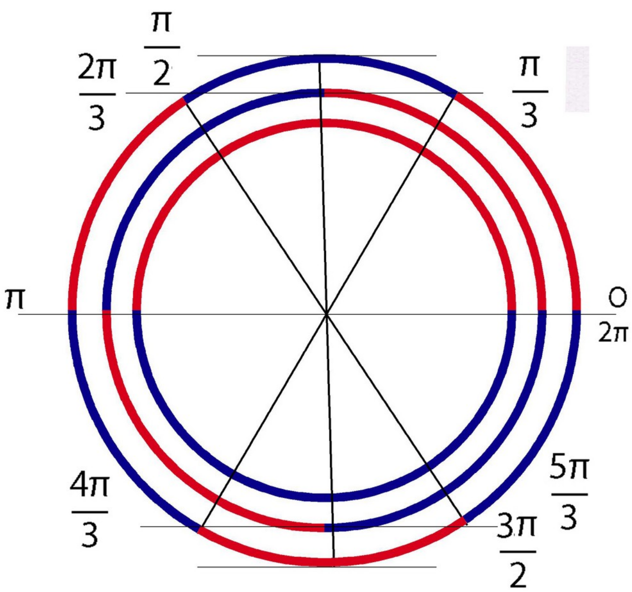


Figure 6

Figure separately the 3 sign status of f(x), g(x), and h(x) on 3 concentric number unit circles. By superimposing, we see that the combined solution set of F(x) > 0 is the 4 open intervals:

(0, Л/3) where the resulting sign is: F(x) = (+)(+)(+) > 0, and
(Л/2, 2Л/3) where → F(x) = (-)(-)(+) > 0, and
(Л, 4Л/3) where → F(x) = (-)(+)(-) > 0, and
(3Л/2, 5Л/3) where → F(x) = (-)(-)(+) > 0

Check. Check the original trig inequality: sin 6x – sin 4x < sin 2x

x = Л/6 → sin (Л) – sin (Л/3) = 0 – sin (Л/3) < sin (Л/3). Proved
x = 5Л/6 → sin (5Л) – sin (4Л/3) = sin (Л) - (-sin Л/3) = 0 + sin (Л/3) > sin (5Л/3). Proved

Note.
- Side transposing during transformation process doesn't affect the original trig inequality.
- Solving these complex trig inequalities by the sign chart is very complicated and confusing.

Example 6. Solve tan x. tan 2x < 0

Solution. F(x) = f(x).g(x) = tan x. tan 2x < 0 (common period: Л)

1. Solve f(x) = tan x = 0 **--**> two end points x = 0, and x = Л.
There is discontinuation at x = Л/2
f(x) > **0** when x varies inside interval (0, Л/2). Color it red, and the rest blue.

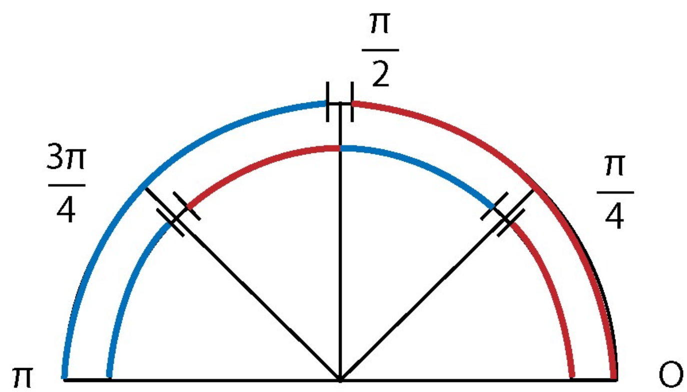


Figure 7

2. Solve g(x) = tan 2x = 0 --> there are 2 end points at x = 0, and x = Л
There are discontinuations at:

2x = Л/2 --> x = Л/4, and 2x = 3Л/2 --> x = 3Л/4

g(x) > 0 when x varies inside the interval **(0, Л/4)**. Color it red and color the other intervals.

By superimposing, we see that the combined solution set are the 2 open intervals:

(Л/4, Л/2) and **(Л/2, 3Л/4)**

Check. Inequality: tan x.tan 2x < 0

x = Л/3 --> tan (Л/3).tan (2Л/3) --> (+).(-) < 0. Proved

x = (2Л/3) → tan (2Л/3).tan (4Л/3) → (-).(+) < 0. Proved

x = Л/6 → tan (Л/6).tan (2Л/6) = tan (Л/6).tan (Л/3) → (+).(+). Not true

x = 5Л/6 --> tan (5Л/6).tan (10Л/6) = tan (5Л/6).tan (5Л/3) --> (-).(-). Not true

REMARKS.

1. A few simple rules for endpoints and arc lengths of simple trig functions in x.
- For f(x) = sin x and f(x) = cos x, and similar, there are 2 endpoints and 2 arc lengths for (0, 2Л)
- For f(x) = sin 2x & f(x) = cos 2x, and similar, there are 4 endpoints and 4 arc lengths for (0, 2Л)
- For f(x) = sin 3x & f(x) = cos 3x, and similar, there are 6 endpoints and 6 arc lengths (0, 2Л)
- For f(x) = tan x there are 1 endpoint, one discontinuation and 3 arc lengths for (0, Л)
- For f(x) = cot x, there are 1 endpoint, one discontinuation, and 3 arc lengths for (Л/2, 3Л/2)

2. The unit circle can be numbered, in radians or degrees, depending on the common period of the trig inequality.
For example:

F(x) = tan x.tan 2x < 0. The unit circle is numbered in interval (0, Л), (or from 0^o to 180^o)
F(x) = sin (x/2).cos 2x < 0. The unit circle is numbered from 0 to 4Л, (or from 0^o to 720^o)
F(x) = sin (x/3).cos x > 0. The unit circle is numbered from 0 to 6Л(or from 0^o to 1080^o)

3. On the trig unit circle, the values of the arc x and the corresponding angle x are exactly equal.

- The **french concept** to select the **arc x** as variable, instead of the **angle x**, makes the whole trigonometric study more convenient, visual, and less absurd.

- The popular expressions such as "Arc tan x", and "Arc sin 3/4" make sense with this concept.

- About the trig functions **tan x** and **cot x**, when they go to infinities, the notion of infinity is understandable as two axis lines going parallel to each other.

- There are 2 axis (AT) and (BU) that define the 2 trig functions tan x and cot x. Solving the trig inequality such as (tan x < a), or similar, is easy and simple by using the tan axis AT. However, solving the trig inequality (sin x/cos x < a) becomes complicated because it is dealing with 2 function variables.

- Solving tan x ≤ a (or ≥ a), or similar, by using the tan function's graph, takes too much time in creating the graph.

- The extended answer ($30^o + 720^o$) can be visually understood as an **arc length** of 30^o plus 2 circumferences. However, the notion of a flat **angle** ($30^o + 720^o$) may be confused as a flat area

- In practice, the rule of endpoints and arc lengths on the unit circle makes the solving process more convenient than the sign chart method.

- Finally, we can use the trig unit circle as a second approach to solve complex trig inequalities by using the bi-unit circle, or triple unit circle.

(This math article is authored by Nghi H Nguyen, **Updated Nov. 24, 2020**)