SOLVING SYSTEMS OF INEQUALITIES IN ONE VARIABLE
BY USING THE DOUBLE NUMBER LINE

(Authored by Nghi H Nguyen – May 06, 1920)

Generalities on solving a quadratic inequalities

Given a quadratic inequality in one variable x under standard form:

$$f(x) = ax^2 + bx + c \leq 0 \quad \text{or} \quad f(x) = ax^2 + bx + c \geq 0$$

Solving this inequality means finding the **solution set** (a set of values) of x that make the inequality true.

Example of quadratic inequalities:

$f(x) = 3x^2 + 7x – 15 < 0$ $\qquad$ $f(x) = - 6x^2 + 11x + 17 < 0$

$f(x) = 5x^2 + 8x – 11 \leq 0$ $\qquad$ $f(x) = 7x^2 – 12x + 13 \geq 0$

The solution set of a quadratic inequality is expressed in the form of **intervals.**

Examples of solution set:

Open intervals:	(3, 14)	(-1, 8)	(-∞, 7)
Half open intervals:	(-2, 3]	(-∞, 4]	(-∞, -5]
Closed intervals:	[-2, 5]	[-0.35, 3.14]	[-1/3, 7/3]

Solving quadratic inequalities by the number line

<u>Example</u>. Solve the inequality: $f(x) = 3x^2 – 8x - 11 < 0$

<u>Solution</u>. First, solve f(x) = 0 to find the two x-intercepts (two real roots). They are x1 = -1 and x2 = 11/3. Figure these real roots on a number line.

```
   -∞               - 1      0                        11/3
<---------------------|--------|---------------------------|------------------------------->
```

Use the origin as the check point. Since f(0) = - 11 < 0, therefor, the origin is located on the true segment, and the solution set is the open interval (-1, 11/3).

Solving a system of two quadratic inequalities by a Double Number Line

Given a system of two quadratic inequalities in one variable:

$f(x) = ax^2 + bx + c \leq 0$ (or ≥ 0)
$g(x) = dx^2 + ex + f \leq 0$ (or ≥ 0)

Solving this system means finding the solution set (set of values) of x that make both inequalities true.

We can use the algebraic method by using a sign chart (sign table). However, using a double number line is simpler and faster.

<u>Example</u>. Solve the system:

$f(x) = x^2 - 2x - 3 < 0$
$g(x) = x^2 - 6x + 5 \leq 0$

<u>Solution</u>. First step, solve the 2 equations:

$f(x) = x^2 - 2x - 3 = 0$ Two real roots: (-1) and (3)
$g(x) = x^2 - 6x + 5 = 0$ Two real roots: (1) and (5)

Next, figure the 2 solution sets of $f(x) < 0$ and $g(x) \leq 0$ on a double number line, with f(x) on the upper line and g(x) on the second line.

```
                -∞          -1       0        1                3               5                 +∞
f(x) < 0        -------------|-------- |--------|---------------|---------------|-----------------------

g(x) ≤ 0        ------------------------------|-----------------------------|-----------------------
```

By superimposing, we see that the segment (1, 3) is the solution set that satisfies both inequalities.

The solution set is the half open interval [1, 3) because 1 is an **end point**

Example. Solve by the number line the system:

$f(x) = 3x^2 – 4x – 7 > 0$
$g(x) = x^2 – x – 6 \geq 0$

Solution. First solve the 2 equations:

$f(x) = 3x^2 – 4x – 7 = 0$ Two real roots: (-1) and (7/3)

$g(x) = x^2 – x – 6 = 0$ Two real roots: (-2) and (3)

Graph on the double number line the solution set of f(x) on the upper line, and the solution set of g(x) on the second line.

```
        -∞                    -1         0                  7/3                  +∞
f(x)    ------------------------|-----------|---------------------|-----------------------------
                    -2                                                  3
g(x)    -------------|---------------------|------------------------------|--------------------
```

By superimposing, we find that the combined solution set is the 2 half open intervals:
(-∞, -2] and [3, +∞). The 2 end points (-2) and (3) belong to the combined solution set.

Example. Solve the system:

$f(x) = 3x^2 – 5x + 2 > 0$
$g(x) = x^2 – 5x - 6 < 0$

Solution. First, solve f(x) = 0 and g(x) = 0

f(x) = 0 Two real roots (1) and (2/3)
g(x) = 0 Two real roots (-1) and (6)

Graph the solution set of f(x) on the first line, and g(x) on the below line.

```
        -∞                        0         1      2/3
f(x)    --------------------------|----------|-------|-------------------------------------------------
                         -1         0          1                                     6
g(x)    -----------------|----------|----------|---------------------------------------|-------------------
```

By superimposing, we see that the combined solution set of the system are the open intervals (-1, 1) and (2/3, 6).

Note. The algebraic method using a sign chart takes longer time and may easily lead to errors/ mistakes

Solving a system of three quadratic inequalities by a Triple Number Line

We can extend this method to solving a system of 3 quadratic inequalities.

<u>Example</u>. Solve the system of 3 quadratic inequalities:

$f(x) = x^2 + x - 2 < 0$
$g(x) = 2x^2 - 7x + 5 > 0$
$h(x) = 2x^2 + 5x - 7 > 0$

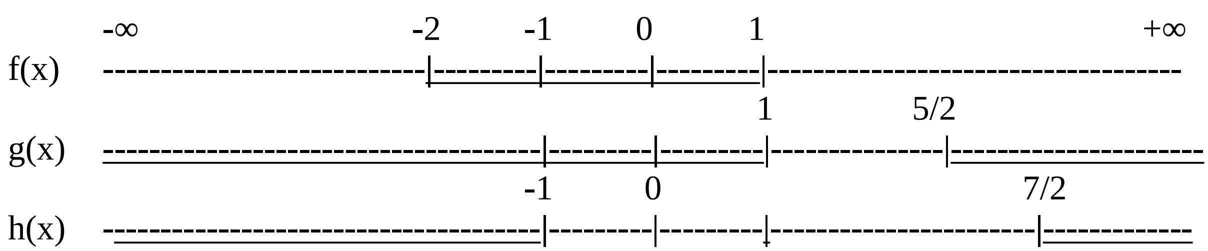
Solution. First step, solve the 3 quadratic equations:

$f(x) = x^2 + x - 2 = 0$ Two real roots: (1) and (-2)

$g(x) = 2x^2 - 7x + 5 = 0$ Two real roots: (1) and (5/2)

$h(x) = 2x^2 - 5x - 7 = 0$ Two real roots: (-1) and (7/2)

Graph a triple number line, and figure all the solution sets on their related line:



By superimposing, we see that the combined solution set is the open interval (-∞, -1), where $f(x) < 0$, and $g(x) > 0$, and $h(x) > 0$.

(This math article was authored by Nghi H Nguyen, the author of the new method “The Transforming Method to solve quadratic equations” – May 06, 2020)