

# SOLVING SYSTEMS OF QUADRATIC INEQUALITIES BY USING THE DOUBLE (or TRIPLE) NUMBER LINE

(Authored by Nghi H Nguyen – Updated on Jan. 01, 2024)

## Generalities on solving quadratic inequalities

Given a quadratic inequality in one variable  $x$  under standard form:

$$f(x) = ax^2 + bx + c \leq 0 \quad \text{or} \quad f(x) = ax^2 + bx + c \geq 0$$

Solving this inequality means finding the **solution set** (a set of values) of  $x$  that makes the inequality true.

Example of quadratic inequalities:

$$f(x) = 3x^2 + 7x - 15 < 0$$

$$f(x) = -6x^2 + 11x + 17 > 0$$

$$f(x) = 5x^2 + 8x - 11 \leq 0$$

$$f(x) = 7x^2 - 12x + 13 \geq 0$$

The solution set of a quadratic inequality is expressed in the form of **intervals**.

Examples of solution set:

Open intervals:             $(3, 14)$                        $(-1, 8)$                        $(-\infty, 7)$

Half open intervals:     $(-2, 3]$                        $(-\infty, 4]$                        $(-\infty, -5]$

Closed intervals:         $[-2, 5]$                        $[-0.35, 3.14]$                        $[-1/3, 7/3]$

## Solving quadratic inequalities by the number line

**Example1.** Solve the inequality:  $f(x) = 3x^2 - 8x - 11 < 0$

Solution. First, solve  $f(x) = 0$  to find the two  $x$ -intercepts (two real roots). They are  $(x_1 = -1)$  and  $(x_2 = 11/3)$ . Plot these real roots on a number line.



Use the origin as the check point. Since  $f(0) = -11 < 0$ , therefore, the origin is located on the true segment, and the solution set is the open interval  $(-1, 11/3)$ .

There is another way to find the answer. Since  $a > 0$ , the graph of  $f(x)$  is an upward parabola. Between the two real roots, the graph is below the axis OX, meaning  $f(x)$  is negative. Therefore, The solution set is the open interval  $(-1, 11/3)$

**Example 2.** Solve:  $-3x^2 < -5x + 2$

Solution. Bring the inequality to standard form:  $f(x) = -3x^2 + 5x - 2 < 0$

First solve  $f(x) = 0$ . Since  $a + b + c = 0$ , one real root is  $x_1 = 1$  and the other  $x_2 = c/a = -2/-3 = 2/3$ . The graph of  $f(x)$  is a downward parabola. So, between the real roots  $f(x)$  is positive. The solution set for  $f(x) < 0$  are the two open intervals  $(-\infty, 2/3)$  and  $(1, +\infty)$



**Example 3.** Solve  $16x^2 \geq 62x - 21$

Solution. Standard form:  $f(x) = 16x^2 - 62x + 21 \geq 0$

First solve  $f(x) = 0$  by the Transforming Method (Google, Studylib.net)

Solve transformed equation:  $f'(x) = x^2 - 62x + 336 = 0$  ( $a \cdot c = 16 \times 21 = 336$ ).

Compose factor pairs of 336. They are: (2, 168), (4, 82), (6, 56).

This last sum is  $6 + 56 = 62 = -b$ . So, the 2 real roots of  $f'(x)$  are 6 and 56.

The 2 real roots of  $f(x) = 0$  are ( $x_1 = 6/a = 6/16 = 3/8$ ) and ( $x_2 = 56/6 = 28/3$ )

Plot these 2 real roots on the number line. The graph is an upward parabola ( $a > 0$ ).

The solution set for  $f(x) \geq 0$  are the 2 half closed intervals  $(-\infty, 3/8]$  and  $[28/3, +\infty)$



## Solving a system of two quadratic inequalities by a Double Number Line

Given a system of two quadratic inequalities in one variable:

$$f(x) = ax^2 + bx + c \leq 0 \quad (\text{or } \geq 0)$$

$$g(x) = dx^2 + ex + f \leq 0 \quad (\text{or } \geq 0)$$

Solving this system means finding the solution set (set of values) of  $x$  that make both inequalities true.

We can use the algebraic method by using a sign chart (sign table). However, using a double number line is simpler and faster.

**Example 4.** Solve the system:

$$f(x) = x^2 - 2x - 3 < 0$$

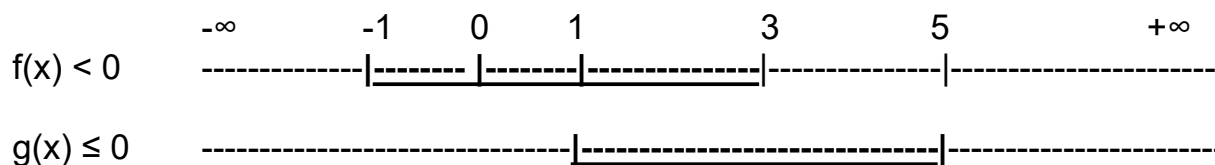
$$g(x) = x^2 - 6x + 5 \leq 0$$

Solution. First step, solve the 2 equations:

$$f(x) = x^2 - 2x - 3 = 0 \quad \text{Two real roots: } (-1) \text{ and } (3)$$

$$g(x) = x^2 - 6x + 5 = 0 \quad \text{Two real roots: } (1) \text{ and } (5)$$

Next, figure the 2 solution sets of  $f(x) < 0$  and  $g(x) \leq 0$  on a double number line, with  $f(x)$  on the upper line and  $g(x)$  on the second line.



**By superimposing**, we see that the segment  $(1, 3)$  is the solution set that satisfies both inequalities.

The solution set is the half open interval  $[1, 3)$  because  $(1)$  is included in the solution set.

**Example 5.** Solve by the double number line the system:

$$f(x) = 3x^2 - 4x - 7 > 0$$

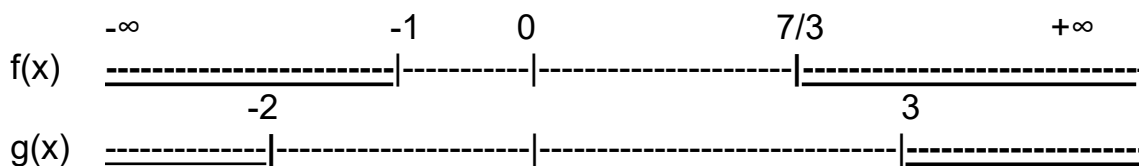
$$g(x) = x^2 - x - 6 \geq 0$$

Solution. First solve the 2 equations:

$$f(x) = 3x^2 - 4x - 7 = 0 \quad \text{Two real roots: } (-1) \text{ and } (7/3)$$

$$g(x) = x^2 - x - 6 = 0 \quad \text{Two real roots: } (-2) \text{ and } (3)$$

Graph on the double number line the solution set of  $f(x)$  on the upper line, and the solution set of  $g(x)$  on the second line.



By superimposing, we see that the combined solution set is the 2 half open intervals:  $(-\infty, -2]$  and  $[3, +\infty)$ .

The 2 end points  $(-2)$  and  $(3)$  belong to the combined solution set.

**Example 6.** Solve the system:

$$f(x) = 3x^2 - 5x + 2 > 0$$

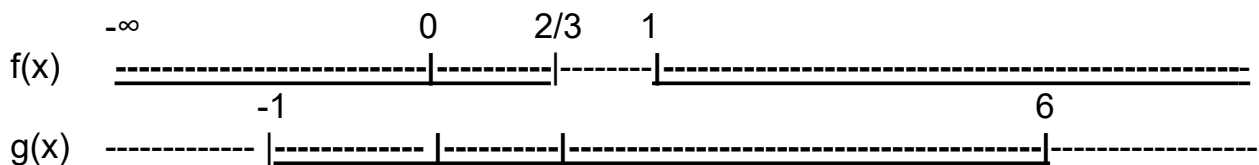
$$g(x) = x^2 - 5x - 6 < 0$$

Solution. First, solve  $f(x) = 0$  and  $g(x) = 0$

$$f(x) = 0 \quad \text{Two real roots } (1) \text{ and } (2/3) \quad \text{Upward parabola}$$

$$g(x) = 0 \quad \text{Two real roots } (-1) \text{ and } (6) \quad \text{Upward parabola}$$

Graph the solution set of  $f(x)$  on the first line, and  $g(x)$  on the below line.



By superimposing, we see that the combined solution set of the system are the open intervals  $(-1, 2/3)$  and  $(1, 6)$ .

### Solving a system of three quadratic inequalities by the Triple Number Line

We can extend this method to solving a system of 3 quadratic inequalities.

**Example 7.** Solve the system of 3 quadratic inequalities:

$$f(x) = x^2 + x - 2 < 0$$

$$g(x) = 2x^2 - 7x + 5 > 0$$

$$h(x) = 2x^2 + 5x - 7 > 0$$

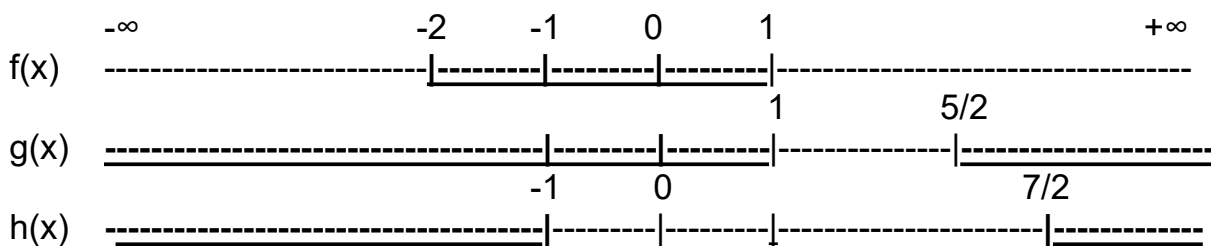
Solution. First step, solve the 3 quadratic equations:

$$f(x) = x^2 + x - 2 = 0 \quad \text{Two real roots: } (1) \text{ and } (-2) \quad \text{Upward parabola}$$

$$g(x) = 2x^2 - 7x + 5 = 0 \quad \text{Two real roots: } (1) \text{ and } (5/2) \quad \text{Upward parabola}$$

$$h(x) = 2x^2 - 5x - 7 = 0 \quad \text{Two real roots: } (-1) \text{ and } (7/2) \quad \text{Upward parabola}$$

Graph a triple number line, and figure all the solution sets on their related line:



By superimposing, we see that the combined solution set is the open interval  $(-2, -1)$ ,

where all 3 inequalities are satisfied:  $f(x) < 0$ , and  $g(x) > 0$ , and  $h(x) > 0$ .

(This math article was authored by Nghi H Nguyen, the author of

- The Transforming Method to solve quadratic equations.
- The Nghi Nguyen Method to solve trig inequalities – Updated on Jan. 01, 2024)