SOLVING SYSTEMS OF QUADRATIC INEQUALITIES
BY USING THE DOUBLE (or TRIPLE) NUMBER LINE

(Authored by Nghi H Nguyen – Updated on Jan. 01, 2024)

Generalities on solving quadratic inequalities

Given a quadratic inequality in one variable x under standard form:

$f(x) = ax^2 + bx + c \leq 0$ or $f(x) = ax^2 + bx + c \geq 0$

Solving this inequality means finding the **solution set** (a set of values) of x that makes the inequality true.

Example of quadratic inequalities:

$f(x) = 3x^2 + 7x – 15 < 0$ $f(x) = - 6x^2 + 11x + 17 > 0$

$f(x) = 5x^2 + 8x – 11 \leq 0$ $f(x) = 7x^2 – 12x + 13 \geq 0$

The solution set of a quadratic inequality is expressed in the form of **intervals.**

Examples of solution set:

Open intervals:	(3, 14)	(-1, 8)	(-∞, 7)
Half open intervals:	(-2, 3]	(-∞, 4]	(-∞, -5]
Closed intervals:	[-2, 5]	[-0.35, 3.14]	[-1/3, 7/3]

Solving quadratic inequalities by the number line

Example1. Solve the inequality: $f(x) = 3x^2 – 8x - 11 < 0$

Solution. First, solve f(x) = 0 to find the two x-intercepts (two real roots). They are (x1 = -1) and (x2 = 11/3). Plot these real roots on a number line.

```
 -∞                  - 1     0                   11/3                 +∞
←---------------------| +++++++++++++++++++++ |---------------------------->
```

Use the origin as the check point. Since $f(0) = -11 < 0$, therefor, the origin is located on the true segment, and the solution set is the open interval (**-1, 11/3**).

There is another way to find the answer. Since $a > 0$, the graph of f(x) is an upward parabola. Between the two real roots, the graph is below the axis OX, meaning f(x) is negative. Therefor, The solution set is the open interval (-1, 11/3)

Example 2. Solve: $-3x^2 < -5x + 2$

Solution. Bring the inequality to standard form: $f(x) = -3x^2 + 5x - 2 < 0$

First solve f(x) = 0. Since a + b + c = 0, one real root is x1 = 1 and the other x2 = c/a = -2/-3 = 2/3. The graph of f(x) is a downward parabola. So, between the real roots f(x) is positive. The solution set for f(x) < 0 are the two open intervals (**-∞, 2/3**) and (**1, +∞**)

-∞ **0** +∞
++++++++|++++++++++++++ 2/3 -------------- 1 **++++++++++++++++++++++**

Example 3. Solve $16x^2 \geq 62x - 21$

Solution. Standard form: $f(x) = 16x^2 - 62x + 21 \geq 0$

First solve f(x) = 0 by the Transforming Method (Google, Studylib.net)

Solve transformed equation: $f'(x) = x^2 - 62x + 336 = 0$ (a.c = 16 x 21 = 336).
Compose factor pairs of 336. They are: (2,168), (4, 82), (6, 56).
This last sum is 6 + 56 = 62 = -b. So, the 2 real roots of f'(x) are 6 and 56.
The 2 real roots of f(x) = 0 are (x1= 6/a = 6/16 = 3/8) and (x2 = 56/6 = 28/3)

Plot these 2 real roots on the number line. The graph is an upward parabola (a > 0).
The solution set for f(x) ≥ 0 are the 2 half closed intervals (**-∞, 3/8**] and [**28/3, +∞**)

-∞ **0** +∞
+++++++++++++++|+++ 3/8 ---------------------------- 28/3 **++++++++++++++**

Solving a system of two quadratic inequalities by a Double Number Line

Given a system of two quadratic inequalities in one variable:

$f(x) = ax^2 + bx + c \leq 0$ (or ≥ 0)
$g(x) = dx^2 + ex + f \leq 0$ (or ≥ 0)

Solving this system means finding the solution set (set of values) of x that make both inequalities true.

We can use the algebraic method by using a sign chart (sign table). However, using a double number line is simpler and faster.

Example 4. Solve the system:

$f(x) = x^2 – 2x – 3 < 0$
$g(x) = x^2 – 6x + 5 \leq 0$

Solution. First step, solve the 2 equations:

$f(x) = x^2 – 2x – 3 = 0$ Two real roots: (-1) and (3)
$g(x) = x^2 – 6x + 5 = 0$ Two real roots: (1) and (5)

Next, figure the 2 solution sets of $f(x) < 0$ and $g(x) \leq 0$ on a double number line, with f(x) on the upper line and g(x) on the second line.

```
             -∞          -1       0      1            3           5              +∞
f(x) < 0     -------------|-------- |--------|--------------|---------------|------------------------

g(x) ≤ 0     ------------------------------|-----------------------------|-----------------------
```

By superimposing, we see that the segment (1, 3) is the solution set that satisfies both inequalities.

The solution set is the half open interval [**1, 3**) because (1) is included in the solution set.

Example 5. Solve by the double number line the system:

$f(x) = 3x^2 – 4x – 7 > 0$
$g(x) = x^2 – x – 6 \geq 0$

Solution. First solve the 2 equations:

$f(x) = 3x^2 – 4x – 7 = 0$ Two real roots: (-1) and (7/3)

$g(x) = x^2 – x – 6 = 0$ Two real roots: (-2) and (3)

Graph on the double number line the solution set of f(x) on the upper line, and the solution set of g(x) on the second line.

```
       -∞                 -1        0                  7/3                +∞
f(x)   ------------------------|-----------|---------------------|----------------------------
                 -2                                            3
g(x)   -------------|---------------------|-----------------------------|--------------------
```

By superimposing, we see that the combined solution set is the 2 half open intervals: (**-∞, -2**] and [**3, +∞**).
The 2 end points (-2) and (3) belong to the combined solution set.

Example 6. Solve the system:

$f(x) = 3x^2 – 5x + 2 > 0$
$g(x) = x^2 – 5x - 6 < 0$

Solution. First, solve f(x) = 0 and g(x) = 0

f(x) = 0 Two real roots (1) and (2/3) Upward parabola
g(x) = 0 Two real roots (-1) and (6) Upward parabola

Graph the solution set of f(x) on the first line, and g(x) on the below line.

```
       -∞                    0        2/3      1
f(x)   --------------------------|----------|--------|--------------------------------------------------
              -1                                                              6
g(x)   ------------- |------------- |----------|---------------------------------------|-------------------
```

By superimposing, we see that the combined solution set of the system are the open intervals (**-1, 2/3**) and (**1, 6**).

Solving a system of three quadratic inequalities by the Triple Number Line

We can extend this method to solving a system of 3 quadratic inequalities.

Example 7. Solve the system of 3 quadratic inequalities:

$f(x) = x^2 + x – 2 < 0$
$g(x) = 2x^2 - 7x + 5 > 0$
$h(x) = 2x^2 + 5x – 7 > 0$

Solution. First step, solve the 3 quadratic equations:

$f(x) = x^2 + x – 2 = 0$ Two real roots: (1) and (-2) Upward parabola

$g(x) = 2x^2 - 7x + 5 = 0$ Two real roots: (1) and (5/2) Upward parabola

$h(x) = 2x^2 - 5x - 7 = 0$ Two real roots: (-1) and (7/2) Upward parabola

Graph a triple number line, and figure all the solution sets on their related line:

```
        -∞                        -2        -1        0         1                                  +∞
f(x)    --------------------------|---------|---------|---------|----------------------------------
                                                                1               5/2
g(x)    ------------------------------------|---------|---------|---------------|--------------------
                                            -1        0                                 7/2
h(x)    ------------------------------------|---------|---------|-----------------------|------------
```

By superimposing, we see that the combined solution set is the open interval (**-2, -1**),

where all 3 inequalities are satisfied: $f(x) < 0$, and $g(x) > 0$, and $h(x) > 0$.

(This math article was authored by Nghi H Nguyen, the author of

- The Transforming Method to solve quadratic equations.
- The Nghi Nguyen Method to solve trig inequalities – Updated on Jan. 01, 2024)