SOLVING QUADRATIC EQUATIONS - COMPARE THE FACTORING “ac METHOD” AND THE NEW DIAGONAL SUM METHOD

By Nghi H. Nguyen

A. GENERALITIES.

When a given quadratic equation can be factored, there are 2 best methods to solve it: the “ac factoring method” (You Tube.com) and the new Diagonal Sum Method (Amazon e-book 2010). This article explains, in general, these two methods, and then compares them through specific examples.

1. The “ac method”.

This method only applies to quadratic equations that can be factored. It is also called “the Box Method” with a little variation (You Tube). Its concept is to make the given quadratic equation be factored into 2 binomials in x by replacing the term (bx) by 2 terms (b1x) and (b2x) that satisfy the 2 conditions:
- 1) the product b1.b2 = ac;
- 2) the sum (b1 + b2) = b.

Example 1. Solve: 5x^2 + 6x – 8.

Solution. Find 2 numbers that the product is (ac = -40) and the sum is b = 6. Proceeding: [(-1, 40),(1, -40),(-2, 20),(2, -20),(-4, 10) OK]. Next, substitute in the equation the term (6x) by the 2 terms (-4x) and (10x) and then put in common factor:

5x^2 – 4x + 10x – 8 = 5x(x + 2) – 4(x + 2) = (x + 2)(5x – 4) = 0

Next, solve the 2 binomials for x:

(x + 2) = 0 → x = -2
(5x – 4) = 0 → x = 4/5

2. The Diagonal Sum method.

Its concept is direct finding the 2 real roots, in the form of 2 fractions, knowing their sum (-b/a) and their product (c/a). It uses 2 rules. The first rule is the Rule of Signs that shows the signs (+ or -) of the 2 real roots before proceeding solving. The second rule is called the Rule for the Diagonal Sum. To know how to use this new Diagonal Sum Method, see the book titled:”New methods for solving quadratic equations and inequalities” (Amazon e-book 2010)

Recall the Rule of Signs.

- If a and c have opposite signs, the 2 real roots have opposite signs.
  Example: the equation 7x^2 – 5x - 12 = 0 has 2 real roots (-1) and (12/7) that have opposite signs

- If a and c have the same sign, both real roots have the same sign. If a and b have same sign, both real roots are negative. If a and b have opposite signs, both real roots are positive.
  Example: the equation ($6x^2 + 13x + 7 = 0$) has 2 real roots (-1) and (-7/6) both negative
  Example: The equation ($5x^2 - 9x + 4 = 0$) has 2 real roots 1 and 4/5, both positive

The Rule of The Diagonal Sum

Given a pair of 2 real roots $\left(\frac{c_1}{a_1}, \frac{c_2}{a_2}\right)$. Their product is: $\frac{c_1c_2}{a_1a_2} = \frac{c}{a}$.

Their sum is: $(c_1/a_1 + c_2/a_2) = (\mathbf{c_1a_2 + c_2a_1})/a_1.a_2 = -b/a$

The sum $(\mathbf{c_1a_2 + c_2a_1})$ is called the Diagonal Sum of a root pair. The Diagonal Sum of a true real root pair must **equal to (–b)**. If it equals (b), the answers are the negative of the pair. If a is negative, the Rule is reversal in signs.

How to proceed with the Diagonal Sum Method.

This new method directly selects the probable root pairs from the (c/a) setup by applying in the same time the Rule of Signs. Then, it uses mental math to calculate their diagonal sum and find the one that equals to –b (or b). If no root pairs equal (-b) or (b), then the equation can't be factored, and the quadratic formula must be used for solving. This new method may be called: "The c/a Method".

Example 2. Solve: $5x^2 + 6x – 8 = 0$.

Solution. The Rule of Signs indicates roots have opposite signs. The (c/a) setup: $\frac{(-1, 8),(-2, 4)}{(1, 5)}$

If the roots have opposite signs, by convention, always put the negative sign (-) in front of the first number of the factors of c. The denominator factors are always kept positive.
First, you can eliminate the pair (-1, 8) because it gives odd number diagonal sums (while b = 6 is even). The remainder (c/a) is: (-2, 4)/(1, 5) that gives 2 probable root pairs: $\left(\frac{-2}{1}, \frac{4}{5}\right)$ and $\left(\frac{-2}{5}, \frac{4}{1}\right)$.
The diagonal of the first pair is: -10 + 4 = -6 = -b. The 2 real roots are: -2 and 4/5.

Example 3. Solve: $5x^2 – 36x + 7 = 0$

Solution. Both real roots are positive. There are 2 probable root pairs: $\left(\frac{1}{1}, \frac{7}{5}\right)$; $\left(\frac{1}{5}, \frac{7}{1}\right)$
The first one can be ignored since 1 is not a real root. The diagonal sum of the second pair is: 35 + 1 = 36 = -b. The 2 real roots are 1/5 and 7.

Example 4. Solve: $8x^2 + 18x + 7$

Solution. Both roots are negative. The c/a setup: (1, 7)/[(1, 8)(2, 4)].

There are 3 probable root pairs: $\left(\frac{-1}{8}, \frac{-7}{8}\right)$; $\left(\frac{-1}{2}, \frac{-7}{4}\right)$, $\left(\frac{-1}{4}, \frac{-7}{2}\right)$

The second diagonal sum is: -4 - 14 = -18 = -b. The 2 real roots are -1/2 and -7/4

Note. Before finding probable root pairs, you may first eliminate the pair (1, 8) because it will give an odd number diagonal sum (while b = 18 is even). The c/a setup remainder (1, 7)/(2, 4) gives 2 probable root pairs (-1/2, -7/4) and (-1/4, -7/2). The first diagonal sum is: -18 = -b. The 2 real roots are -1/2 and -7/4.

B. SOLVING QUADRTIC EQUATIONS IN DIFFERENT CASES

a. CASE 1. When a = 1 – Solving the quadratic equation type: $x^2 + bx + c = 0$.

In this case, solving results in finding 2 numbers knowing their sum (-b) and their product c.

<u>Example 5</u>. Solve: $x^2 - 9x - 102 = 0$.

1. Solving by the Diagonal Sum Method. Roots have opposite signs. Write factor pairs of c = -102 and, in the same time, apply the Rule of Signs: (-1, 102), (-2, 51), (-3, 34), (-6, 17)…Stop! This sum is 17 – 6 = 9 = -b. The 2 real roots are -6 and 17.

2. Solving by the "ac method". Find 2 numbers that the product is: ac = - 102 and the sum is b = -9. Proceed: [(-1, 102), (1, -102), (-2, 51),(51, -2),(-3, 34),(3, -34),(-6, 17), (6, -17), OK]. Next, replace the term (-9x) by two terms (6x) and (-17x):

$$x^2 - 9x - 102 = x^2 + 6x - 17x - 102 = 0$$
$$x(x - 17) + 6(x - 17) = 0$$
$$(x - 17)(x + 6) = 0$$

Solve the 2 binomials for x:

$$(x + 17) = 0 \rightarrow x = -17$$
$$(x - 6) = 0 \rightarrow x = 6$$

3. Remark. In this case, solving by the Diagonal Sum method is simpler and doesn't need factoring. The Rule of Signs, that shows the signs of the 2 real roots before proceeding solving, reduces in half the number of permutations (or test cases). In addition, it saves the time used to solve the 2 binomials for x.

<u>Example 6.</u> Solve: $-x^2 + 28x - 96 = 0$.

1. **<u>Solving by diagonal sum method.</u>** The Rule of signs indicates both roots are positive.

Write factor-pairs of ac = 96 after applying the Rule of Signs: (1, 96), (2, 48), (3, 32), (4, 24)…This sum is: 4 + 24 = 28 = b. According to the Diagonal Sum Rule, when a is negative, the 2 real roots are 4 and 24.

2. **Solving by the "ac method".** Find 2 numbers with product ac = 96, and sum = 28. Proceeding: [(-1, -96),(1, 96),(-2, -48),(2, 48),(-3, -32),(3, 32),(-4, 24)(4, 24) OK]. Replace (28x) by (4x) and (24x) in the equation:

$$-x^2 + 28x - 96 = -x^2 + 4x + 24x - 96 = 0$$
$$-x(x - 4) + 24(x - 4) = (x - 4)(24 - x) = 0$$

Next, solve the 2 binomials:

$$x - 4 = 0 \rightarrow x = 4$$
$$24 - x = 0 \rightarrow x = 24$$

b. CASE 2. When a and c are prime numbers.

Example 7. Solve: $7x^2 - 76x - 11 = 0$.

1..**Diagonal Sum Method.** Roots have opposite signs. Both a and c are prime. There is unique probable root-pair: $\left(\frac{-1}{7}, \frac{11}{1}\right)$, since (-1) is not a real root.

Its diagonal sum is 77 - 1 = 76 = -b. The 2 real roots are -1/7 and 11.

2. **The factoring "ac method".** Find 2 numbers that the product is: ac = -77, and the sum b = 76. Proceeding: [(-1, 77),(-, -77)]. Next, replace the term (-76x) by the 2 terms (1x) and (-77x).

$$7x^2 + x - 77x - 11 = 0$$
$$7x(x - 11) + (x - 11) = (x - 11)(7x + 1) = 0$$

Solve the 2 binomials for x

$$(x - 11) = 0 \rightarrow x = 11$$
$$(7x + 1) = 0 \rightarrow x = -1/7$$

3. **Remark**. In this case, solving by the Diagonal Sum Method is faster, since there is only one diagonal sum to find.

c. CASE 3. When a and c are small numbers and may contain themselves one or 2 factors

Example 8. Solve: $8x^2 - 22x - 13 = 0$.

1.**The Diagonal Sum method**. Roots have opposite signs. Write the (c/a) setup:

(-1, 13)/[(1, 8),(2, 4)]. Eliminate the pair (1, 8) because it will give odd-number diagonal sum (while b is even).
It remains 2 probable root-pairs: $(\frac{-1}{2}, \frac{13}{4})$ and $(\frac{-1}{4}, \frac{13}{2})$.
The first diagonal sum is: -4 + 26 = 22 = -b. The 2 real roots are: -1/2 and 13/4.

2.The factoring "ac method". Find 2 numbers, that their product is ac = -104, and their sum is -22. Proceeding: [(-1, 104),(1, -104),(-2, 52),(2, -52), (-4, 26),(4, -26)]. Replace the term -22x by the 2 terms (4x) and (-26x).

$$8x^2 + 4x - 26x - 13 = 0.$$
$$4x(2x + 1) - 13(2x + 1) = 0$$
$$2x + 1) (4x - 13) = 0$$

Next, solve the 2 binomials:

$$2x + 1 = 0 \rightarrow x = -1/2$$
$$4x - 13 = 0 \rightarrow. X = 13/4$$

d. CASE 4. When a and c are large numbers and may contain themselves a few factors

These cases are considered complicated because there are many permutations involved. The Diagonal Sum Method may transform a complicated multiple steps solving process into a simplified one by doing a few elimination operations.

Example 9. Solve: $12x^2 + 5x - 72 = 0$.

1.**<u>Diagonal Sum method</u>**. Roots have opposite signs. In this case, do not directly write down the probable root-pairs because there are too many of them. First, create the (c/a) setup, with all factor pairs of c and of a, and in the same time apply the Rule of Signs:

c = - 72 -→	(-1, 72)(-2, 36)(-3, 24)(-4, 18)(-6, 12)(-8, 9).
a = 12	(1, 12) (2, 6) (3, 4)

Before solving, look to eliminate the pairs that do not fit. First, eliminate the pairs: (-2, 36), (-4, 18),(-6, 12) from the numerator and the pair (2, 6) from the denominator because they give even- number diagonal sums (while b = 5 is odd number).
Next, eliminate the pairs (-1, 72) (-3, 24)/(1, 12) because they give large diagonal sums (while b= 5). The remainder (c/a) setup is: (-8, 9)/ (3, 4). This gives 2 probable root-pairs: (-8/3, 9/4) and (-8/4, 9/3). The diagonal sum of the first pair is: 27 – 32 = -5 = -b. The 2 real roots are: -8/3 and 9/4.

2..**<u>The factoring “ac method”.</u>** In these cases, solving becomes inconvenient because the product ac is a large number. Find 2 numbers: Product ac = -864. Sum: b = 5. Proceeding: [(-1, 864),(1, -864),(-2, 432),(2, -432),(-3, 288),…………(-18, 48),(18, -48),(-24, 36),(24, -36),(- 32, 27),(32, -27) OK]. Next, replace the term 5x by the 2 terms -27x and 32x.

$$12x^2 – 27x + 32x – 72 = 0.$$
$$3x(4x – 9) + 8(4x - 9) = 0$$
$$(4x – 9)(3x + 8) = 0$$

Next, solve the 2 binomials:

$$4x – 9 = 0 \rightarrow x = 9/4.$$
$$3x + 8 = 0 \rightarrow x = -8/3.$$

Example 10. Solve: 24x^2 + 59x + 36 = 0.

1. **Diagonal Sum method.** Both roots are negative. The (c/a) setup:

<u>(-1, -36)(-2, -18)(-3, -12)(-4, -9)(-6, -6)</u>
(1, 24)(2, 12) (3, 8) (4, 6)

Before proceeding, eliminate the pairs (-2, -18),(-6, 6)/(2, 12),(4, 6) because they give even-number diagonal sums (while b = 59 is odd).
Also, eliminate the pairs: (-1, -36)(-3, -12)/(1, 24) because they give large diagonal sums, as compared to b = 59. The remainder (c/a) is: <u>(-4, -9)</u>
(3, 8)

This gives 2 probable root-pairs: $\left(\frac{-4}{3}, \frac{-9}{8}\right)$ and $\left(\frac{-4}{8}, \frac{-9}{3}\right)$. The diagonal sum of the first pair is: -32 – 27 = -59 = -b. The 2 real roots are -4/3 and -9/8.

2.**The “ac method”.** Find 2 numbers: product ac = 864; sum b = 59. Proceeding [(-1, -864)(1, 864),(-2 , -432),(2, 432),(-4, -216),(4, 216), ………..(-18, - 48)(18, 48)(-24, -36) (24, 36),(-27, -32)(27, 32)]. Since the product ac = 864 is too large, the proceeding takes too much time to complete.
Next, replace term (59x) by the 2 term(27x) and (32x).

$$24x^2 + 27x + 32x + 36 = 0.$$
$$8x(3x + 4) + 9(3x + 4) = 0$$
$$(3x + 4)(8x + 9) = 0$$
$$3x + 4 = 0 \rightarrow x = -4/3$$
$$8x + 9 = 0 \rightarrow x = -9/8.$$

A. CONCLUSION AND REMARKS.

1. Both methods deserve to be studied because they provide students with opportunities to improve math skills and logical thinking that are the ultimate goals of learning math.

2. When the constants a and c are large numbers and may contain themselves many factors, then students are advised to use the quadratic formula for solving.
   However, either performing mentally or by calculator, remember to always proceed solving in 2 steps.
   First step, compute the Discriminant D = b^2 – 4ac. If the given quadratic equation can be factored, then D must be a perfect square.
   Second step, compute algebraically the rest of the formula with the value of D's square root that should be a whole number. Make sure that the 2 real roots be in the form of 2 fractions and not in decimals.
   If calculators are not allowed, solving by the Diagonal Sum Method may be faster and better.

3. Best methods to solve quadratic equations. The quadratic formula is obviously the best choice to solve a quadratic equation in standard form ax^2 + bx + c = 0, especially when calculators are allowed. However the ultimate goal of math learning isn't only to solve equations with calculators. The math learning process wants students to learn solving quadratic equations by a few other methods in order to improve their math skills and their logical thinking. Although 99% (?) of the quadratic equations in real life are not factorable, many math equations in books/tests/exams are intentionally setup so that students must solve them by the factoring method. So far, the 2 best methods to solve factorable quadratic equations are the "ac method" and the new Diagonal Sum Method.

   (This article was written by Nghi H Nguyen, the co-author of the new Diagonal Sum Method for solving quadratic equations)