SOLVING QUADRATIC EQUATIONS BY THE DIAGONAL SUM METHOD

A quadratic equation in one variable has as standard form: ax^2 + bx + c = 0. Solving it means finding the values of x that make the equation true.

Beyond the 4 common existing solving methods (quadratic formula, factoring, completing the square, and graphing), there is a new solving method, called Diagonal Sum Method, recently presented in book titled ”New methods for solving quadratic equations and inequalities” (Amazon e-book 2010). This new solving method can directly obtain the 2 real roots without having to factor the equation.

It is a trial and error method, same as the factoring one, but it reduces the number of permutations in half by using a Rule of Signs for Real Roots. It is fast, convenient and is applicable whenever the equation is factorable. If this method fails to get the answer, then the equation can't be factored, and consequently, the quadratic formula must be used. There is a new **improved quadratic formula**, called the quadratic formula in graphic form, developed in the above mentioned book, that is easier to understand and remember, since students can relate the formula to the x-intersects of the parabola graph of the quadratic function.

Innovative concept of the Diagonal Sum Method.

Direct finding 2 real roots, in the form of 2 fractions, knowing their sum (-b/a) and their product (c/a).

Recall the Rule of Signs for Real Roots.

It is needed to know in advance the sign status (+ or -) of the 2 real roots in order to reduce the number of permutations, or test cases. The Rule of Signs states as follows:

- If a and c have opposite signs, the 2 real roots have opposite signs.
  Example: The equation 5x^2 – 14x – 3 = 0 has 2 real roots with opposite signs
- If a and c have same sign, the real roots have same sign and it can be further possible to know if both real roots are positive or negative
  a. If a and b have opposite signs, both real roots are positive.
  Example: The equation x^2 – 39x + 108 = 0 has 2 real roots both positive
  b. If a and b have same sign, both roots are negative.
  Example: The equation x^2 + 27x + 50 = 0 has 2 real roots both negative

More examples of using the Rule of Signs:

The equation -6x^2 + 7x + 20 = 0 has 2 real roots that have opposite signs
The equation 12x^2 – 113x + 253 = 0 has 2 real roots, both positive.
The equation 21x^2 + 50x + 24 = 0 has 2 real roots, both negative.

The diagonal sum of a pair of 2 real roots.

Given a pair of 2 real roots ($\frac{c1}{a1}$, $\frac{c2}{a2}$).

Their product equals to ($\frac{c}{a}$), meaning c1.c2 = c, and a1.a2 = a.

The numerators of a probable root pair come from a factor pair of c. The denominators come from a factor pair of a.

The sum of the 2 real roots is equal to ($\frac{-b}{a}$).

The sum: $\frac{c1}{a1} + \frac{c2}{a2} = \frac{(c1a2 + c2a1)}{a1.a2} = \frac{-b}{a}$

The sum (c1a2 + c2a1) is called the **diagonal sum** of a root-pair.

Rule for the Diagonal Sum.

The diagonal sum of a **true** root-pair must equal to (-b). If it equals to b, then it is the negative of the solution. If the constant a is negative, the above rule is reversal in sign, meaning the diagonal sum of a real root pair must equal to (b).

Tips for quick solving special quadratic equations.

- **Tip 1**: When a + b + c = 0, one real root is 1 and the other is (c/a)
  Example: The equation: 3x^2 – 5x + 2 = 0 has 2 real roots: 1 and 2/3
- **Tip 2**: When a – b + c = 0, one real root is -1 and the other is (-c/a)
  The equation: 5x^2 – 2x – 7 = 0 has 2 real roots: -1 and 7/5

SOLVING QUADRATIC EQUATIONS IN VARIOUS CASES USING THE DIAGONAL SUM METHOD.

Depending on the values of the constants a and c, solving quadratic equations may be simple or complicated.

A. When a = 1. Solving the quadratic equation type: x^2 + bx + c = 0.

In this special case, the diagonal sum becomes the sum of the 2 real roots. Solving results in finding 2 numbers knowing their sum (–b) and their product (c). Solving this type of quadratic equations by the Diagonal Sum Method is simple, fast and doesn't need factoring.

Example 1. Solve: x^2 - 21x – 72 = 0.

Solution. The Rule of Signs indicates the roots have opposite signs. Write the factor-pairs of c = -72 and apply in the same time the Rule of Signs: (-1, 72) (-2, 36) (-3, 24)…Stop here! The sum of the 2 roots in this pair is 21 = -b. The 2 real roots are: -3 and 24.

Note. There are factor pairs with opposite signs: (1, -72) (2, -36)…but they can be ignored since they give the corresponding opposite diagonal sums. By convention, always put the negative sign in front of the first number of the pair.

Example 2. Solve: $-x^2 - 26x + 56 = 0$.

Solution. Roots have opposite signs. The constant a is negative. Write factor pairs of ac = -56: (-1, 56) (-2, 28)…Stop here! This sum is 26 = -b. According to the Rule for the Diagonal Sum, when **a** is negative the sum must equal to (b). The true root-pair is the negative of the pair (-2, 28). The 2 real roots are: 2 and -28.

Example 3. Solve: $x^2 + 27x + 50 = 0$.

Solution. Both roots are negative. Write factor pairs of c = 50 and apply the Rule of Signs: (-1, -50) (-2, -25)…Stop here! This sum is -27 = -b. The 2 real roots are -2 and -25.

Example 4. Solve $x^2 - 39x + 108 = 0$.

Solution. Both roots are positive. Write factor pairs of c and apply the Rule of Signs.
They are: (1, 108) (2, 54) (3, 36)…Stop! This sum is 3 + 36 = 39 = -b.
The 2 real roots are: 3 and 36.

B. When a and c are prime numbers.

The Diagonal Sum Method directly selects the probable root-pairs from the values of the constants a and c and in the same time applies the Rule of Signs to these pairs.
When a and c are both **prime** numbers, the number of probable root-pairs is usually limited to one, except the cases of Tip 1 (or Tip 2).

Example 5. Solve: $7x^2 + 90x - 13 = 0$.

Solution. Roots have opposite signs. Select the probable root-pairs. The numerator contains the unique factor pair of c (-1, 13). The denominator contains the unique factor pair of a (1, 7). Permutation should be done to the denominator that keeps always positive.

Unique root-pair: $\left(-\frac{1}{7}, \frac{13}{1}\right)$ The diagonal sum: -1 + 91 = 90 = b

Since the diagonal equals to b, the answers are the negative of this pair: 1/7 and -13.

Note 1: The other root-pair (-1/1, 13/7) can be ignored since -1 is not a real root (Tip 2)

Note 2: When the roots have opposite signs, by convention, always put the negative sign (-) in front of the first number of the pair. The denominators are always kept positive.
Examples of probable root pairs: $\left(\frac{-1}{7}, \frac{13}{1}\right)$; $\left(\frac{-1}{3}, \frac{19}{4}\right)$

Example 6. Solve: 17x^2 + 324x + 19 = 0.

Solution. Both roots are negative. The constants a, c are both prime numbers.

Since 1 is not a real root (Tip 1), there is unique root pair: $\left(\frac{-1}{17}, \frac{-19}{1}\right)$.

Its diagonal sum is: -323 -1 = -324 = -b. The 2 real roots are: -1/17 and -19.

C. When a and c are small numbers

In general, the Diagonal Sum Method proceeds by considering the c/a setup. The numerator contains a few factor pairs of c, and the denominator contains a few factor pairs of a.
In case when a and c are small numbers and may contain themselves one (or 2) factors, you may write down all probable root pairs, then use mental math to calculate their diagonal sums and find the one that fits. Stop calculation when one diagonal sum matches –b (or b).
Or you may first simplify the c/a setup by eliminating the pairs that don't fit. The remainder c/a will lead to just a few probable root pairs.

Example 7. Solve: 7x^2 – 57x + 8 = 0.

Solution. Both roots are positive. The constant c = 8 has 2 factor pairs: (1, 8), (2, 4).
All probable root pairs: $\left(\frac{1}{7}, \frac{8}{1}\right)$ $\left(\frac{2}{1}, \frac{4}{7}\right)$ $\left(\frac{2}{7}, \frac{4}{1}\right)$
Diagonal sums: (57 = -b) Answers: 1/7 and 8.

NOTE. You may simplify the c/a setup: (1, 8)(2, 4)/(1, 7). Eliminate the pair (2, 4) because it gives even number diagonal sums (while b = -57 is odd). The remainder c/a: (1, 8)/(1, 7) leads to unique probable root pair: (1/7, 8/1). Its diagonal sum is 76 + 1 = 57 = -b. The 2 real roots are 1/7 and 8.

Example 8. Solve: 6x^2 – 19x – 11 = 0.

Solution. Roots have opposite signs. The constant a = 6 has 2 factor-pairs: (1, 6), (2, 3)

Probable root pairs: $\left(\frac{-1}{6}, \frac{11}{1}\right)$ $\left(\frac{-1}{2}, \frac{11}{3}\right)$ $\left(\frac{-1}{3}, \frac{11}{2}\right)$
Diagonal sums: (66 – 1 = 65) (19 = -b)
The 2 real roots are -1/2 and 11/3

Note 1: There are other opposite root-pairs: $(\frac{1}{6}, \frac{-11}{1})$, $(\frac{1}{2}, \frac{-11}{3})$,… but they can be ignored since they give the same values of diagonal sums in opposite sign.

Example 9. Solve: -11x^2 + 26x + 21 = 0

Solution. Roots have opposite signs, and a is negative. Probable root pairs:

	$(\frac{-7}{11}, \frac{3}{1})$;	$(\frac{-7}{1}, \frac{3}{11})$	
DS	(26 = b)		Answer: -7/11 and 3

D. When the constant a is negative.

Option 1. When a is negative, you can change the equation to the one with constant a positive by putting (-1) in common factor.

Example 10. Solve: – 11x^2 + 26x + 21 = 0 (1)

Solution. Make the constant a positive by adding the factor (-1):

(-1)(11x^2 – 26x - 21) = 0 (2)

Now, solve the equation (2) as usual. Remember that both Equation (1) and Equation (2) have the same root-pair.

Probable root-pairs:	$(\frac{-3}{1}, \frac{7}{11})$	$(\frac{-3}{11}, \frac{7}{1})$
DS	(-26 = b)	

Since the diagonal sum equals to b when a is positive, then the answers are opposite to the fractions in this pair, they are: 3 and -7/11

Option 2. When the constant a is negative, you can directly solve the equation by the new method, but you must reverse the matching rule: the diagonal sum should equal to (b), instead of (-b).

Example 11. Solve: - 7x^2 + 34x + 5 = 0

Solution. Real roots have opposite signs. Constant a is negative. Probable root-pairs:

	$(\frac{-1}{1}, \frac{5}{7})$	$(\frac{-1}{7}, \frac{5}{1})$	
DS	(Ignore)	(34 = b)	Answers: -1/7 and 5

Example 12. Solve: - 5x^2 – 69x + 14 = 0

Solution. Roots have opposite signs. Probable root pairs:

	$\left(\frac{-2}{1}, \frac{7}{5}\right)$	$\left(\frac{-2}{5}, \frac{7}{1}\right)$	$\left(\frac{-1}{5}, \frac{14}{1}\right)$	
DS			(69 = -b)	Answers: 1/5 and -14

Example 13. Solve – 8x^2 + 30x – 7 = 0.

Solution. Both real roots are positive.

	$\left(\frac{1}{8}, \frac{7}{1}\right)$	$\left(\frac{1}{2}, \frac{7}{4}\right)$	$\left(\frac{1}{4}, \frac{7}{2}\right)$	
DS			(30 = b)	Answers: 1/4 and 7/2

E. When a and c are large numbers and contain themselves many factors.

These cases are considered complicated because they involve many permutations in the solving process. In these cases, to make sure that no root-pair is omitted, it is advised to write all the factor pairs in the (c/a) setup. In this setup, the numerator contains all factor-pairs of c (after applying The Rule of Signs). The denominator contains all factor pairs of a (after applying the Rule of Signs). If the given equation can be factored, one of the (c/a) combinations should lead to the 2 real roots.
Next, the Diagonal Sum Method transforms a multiple steps solving process into a simplified one by doing a few operations of elimination.

Example 14. Solve: 8x^2 + 13x – 6 = 0.

Solution. Roots have opposite signs. Write the (c/a) setup and apply in the same time the Rule of Signs:

Numerator. Factor pairs of c = -6: (-1, 6) (-2, 3)
Denominator. Factor pairs of a = 8: (1, 8) (2, 4)

Now, you can use mental math, or a calculator, to calculate all diagonal sums and find the one that fits.
Or, you can proceed to eliminate the pairs that don't fit. Eliminate the pair (2, 4) because if combined to any pair of c, it will give even-number diagonal sums (while b is odd). Then, eliminate the pair (-1, 6) because it gives larger diagonal sum than b = 13. The remainder (c/a) $\frac{(-2,\ 3)}{(1,\ 8)}$ leads to 2 probable root pairs: $\left(\frac{-2}{1}, \frac{3}{8}\right)$ and $\left(\frac{-2}{8}, \frac{3}{1}\right)$.
The diagonal sum of the first pair is: -16 + 3 = -13 = -b. The real roots are: -2 and 3/8.

Example 15. Solve: 45x^2 – 74x – 55 = 0.

Solution. Roots have opposite signs. Write the (c/a) setup and apply the Rule of Signs:

Numerator. Factor pairs of c = -55: (-1, 55) (-5, 11)
Denominator. Factor pairs of a = 45: (1, 45) (3, 15) (5, 9)

You may eliminate pairs that don't fit. First, eliminate the pairs
(-1, 55) because they give large diagonal sums, as compared to b = -74.
(1, 45), (3, 15)
The remainder c/a setup: (-5, 11), gives the unique root pair (-5 , 11)
(5, 9) 9 5
Its diagonal sum is -25 + 99 = 74 = -b. The 2 real roots are: -5/9 and 11/5.

Example 16. Solve: 12x^2 – 272x + 45 = 0.

Solution. Both roots are positive. Write the (c/a) setup and apply the Rule of Signs:

Numerator. Factor-pairs of c = 45: (1, 45) (3, 15) (5, 9)
Denominator. Factor-pairs of a = 12; (1, 12) (2, 6) (3, 4)

First, eliminate the pairs (1, 12) and (3, 4) because, combined with any pair of c, they give odd-number diagonal sums while b is an even number. Next, look for a large-number diagonal-sum (-272).
The fitted c/a setup should be (1, 45), that gives the 2 real roots: 1/6 and 45/2.
(2, 6)

Example 17. Solve: 40x^2 – 111x + 36 = 0.

Solution. Both roots are positive. Write the (c/a) setup and apply the Rule of Signs:

Numerator. (1, 36) (2, 18) (3, 12) (6, 6)
Denominator. (1, 40) (2, 20) (4, 10)(5, 8)

First, eliminate the pairs (2, 18)(6, 16)/(2, 20)(4, 10) since they give even-number diagonal sums (while b is odd). Then, eliminate pairs (1, 36)/(1, 40) because they give large diagonal sums, while b = -111. The remainder (c/a) is: (3, 12)/(5, 8) that leads to 2 probable root pairs: (3 , 12) and (3 , 12).
5 8 8 5
The second pair gives as diagonal sum 15 + 96 = 111 = -b. The 2 real roots that are: 3/8 and 12/5.

Example 18. Solve: 12x^2 + 5x – 72 = 0.
Solution. Roots have opposite signs. Write the (c/a) setup and apply the Rule of Signs:

Numerator: (-1, 72) (-2, 36) (-3, 24) (-4, 18) (-6, 12) (-8, 9)
Denominator: (1, 12) (2, 6) (3, 4)

First, eliminate the pairs (-2, 36) (-4, 18) (-6, 12)/(2, 6) since they give even number diagonal sums (while b is odd). Then, eliminate pairs (-1, 72)(-3, 24) /(1, 12) because they give large-number diagonal sums (while b = 5). The remainder (c/a) is

(-8, 9), that leads to two probable root pairs: (-8 , 9) and (-8 , 9).
 3 4   3 4   4 3
The first pair's diagonal sum is: (27 – 32 = -5 = -b). The 2 real roots are: -8/3 and 9/4.

Example 19. Solve: 24x^2 + 59x + 36 = 0.

Solution. Both roots are negative. Write the (c/a) setup and apply the Rule of Signs:

Numerator: (-1, -36) (-2, -18) (-3, -12) (-4, -9) (-6, -6)
Denominator: (1, 24) (2, 12) (3, 8) (4, 6)

First, eliminate pairs (-2, -18)(-6, -6)/ (2, 12)(4, 6) because they give even-number-diagonal-sums (while b is odd). Then, eliminate pairs (-1, -36)(-2, -18) /(1, 24) since they give large diagonal-sums (while b = -59).
The remainder (c/a) is: (-4, -9), that gives 2 real root pairs: (-4 , 9) and (-4 , 9)
(3, 8)   3 8   8 3
This first pair gives two real roots: -4/3 and -9/8.

Notes: Here are some common practices to eliminate factor pairs that don't fit:

1. Eliminate the pairs that give extreme diagonal sums (too large or too small) as compared with the value of the constant b.
2. If b is an **odd** number, eliminate the factor pairs, such as (-2, 4), (4, 10), (-4, -6),…, because they give even-number diagonal sums.
3. If b is an **odd** number, eliminate the c/a that contains all 4 odd numbers, such as (1, 3)/(5, 7); (3, 7)/(1, 9)…, because they give even number diagonal sums.
4. If b is an **even** number, eliminate the c/a that have one even number and 3 odd numbers, such as (1, **4**)/(3, 5); (3, 5)/3, **8**); (**2**, 7)/(5, 9)…, because they give odd number diagonal sums.

Example 19. Solve: 8x^2 – 35x + 12 = 0.

Solution: Both real roots are positive. Write the (c/a) setup: (1, 12)(2, 6)(3, 4)
(1, 8)(2, 4)
First eliminate the pairs (2, 6) and (2, 4) since they give even number diagonal sums.
Then, eliminate the c/a: (1, 12)/(1, 8) since it gives large diagonal sum (while b is 35).
The remainder (c/a): (3, 4) leads to probable root pairs: (3 , 4) and (3 , 4)
(1, 8)   1 8   8 1

The second pair gives as diagonal sum 35 = -b. The real roots are 3/8 and 4.

5. **5.** If b is an even number, eliminate pairs, such as (-1, 4); (3, 6); (1, -8)….because they will give odd number diagonal sums.

Example . Solve: $8x^2 + 2x - 15 = 0$

Solution. Roots have opposite signs. Write the (c/a) setup: $\frac{(-1, 15)(-3, 5)}{(1, 8)(2, 4)}$

First, eliminate the pair (1, 8) because, combined with any pair from c, they give odd-number diagonal sums (while b is even). Next, eliminate the pair (-1, 15) because it gives large diagonal sums (while b = 2).

The remainder c/a setup: $\frac{(-3, 5)}{(2, 4)}$ will lead to two probable root pairs $\left(\frac{-3}{2}, \frac{5}{4}\right)$ and $\left(\frac{-3}{4}, \frac{5}{2}\right)$.

The diagonal sum of the first pair is (-2 = -b). The 2 real roots are -3/2 and 5/4.

Comments.

1. So far, the systematic "ac Method", or the similar "Box Method" (youtube.com) are the ones that generally solve quadratic equations in standard form. Both methods proceed starting from the product (ac). When the constants a and c are large number and may contain themselves many factors, both methods become complicated because the (ac) product is too big to handle.

2. The Diagonal Sum method proceeds by basing on the (c/a) setup. Therefore, it may be called: "The c/a Method". This new method can simplify a multiple steps process into a simplified one by doing a few elimination operations.

3. When c and a are both large numbers and may contain themselves many factors, it is advised that you use the quadratic formula for solving, if calculators are allowed. However, remember to solve these equations in 2 steps. In first step, compute by calculator the Discriminant $D = b^2 - 4ac$. If the given quadratic equation can be factored, D must be a perfect square and the square root of D must be a whole number. In second step, algebraically calculate the 2 real roots by the quadratic formula. Make sure that the 2 real roots be in the form of 2 fractions and not in decimals.
   If calculators are not allowed, then the Diagonal Sum Method may proceed faster and better.

(This article was written by Nghi H. Nguyen, the co-author of the new Diagonal Sum Method for solving quadratic equations)