SOLVING TRIG INEQUALITIES BY NGHI NGUYEN METHOD

- Exercises (Part II) -

(Authored by Nghi H Nguyen – Feb 01, 2021)

GENERALITIES

To solve a complex trig inequality F(x) by Nghi H Nguyen's method (Google Search), we must transform the inequality F(x) into many basic trig functions (or similar) in the form:

$F(x) = f(x).g(x) \leq 0$ (or ≥ 0) or $F(x) = f(x).g(x).h(x) \leq 0$ (or ≥ 0)

Transformation means include using trig identities and common algebraic manipulations.

TRANSFORMATION USING COMMON ALGEBRAIC MANIPULATIONS AND TRIG IDENTITIES

Exercise 1. Solve $F(x) = \sin^2 x - \cos^2 x > 0$

Solution. Use the algebraic identity: $a^2 - b^2 = (a - b)(a + b)$ to transform the inequality.
$F(x) = f(x).g(x) = (\sin x - \cos x).(\sin x + \cos x)$

a. Solve $f(x) = \sin x - \cos x = \sqrt{2}.\sin (x - \pi/4) = 0$. There are 3 solutions:

$x - \pi/4 = 0$, that gives $x = \pi/4$
$x - \pi/4 = \pi$, that gives $x = \pi + \pi/4 = 5\pi/4$
$x - \pi/4 = 2\pi$, that gives $x = 2\pi + \pi/4 = 9\pi/4$ or $x = \pi/4$

There are 2 end points at $(\pi/4)$ and $(5\pi/4)$ and 2 arc lengths. To find the sign status of f(x), select point $(\pi/2)$ as check point. We have: $f(\pi/2) = \sin (\pi/2) - \cos (\pi/2) = 1 - 0 = 1 > 0$. Then, f(x) is positive (> 0) inside the interval $(\pi/4, 5\pi/4)$. Color it red and color the other arc length blue.

b. Solve $g(x) = \sin x + \cos x = \sqrt{2}.\sin (x + \pi/4) = 0$. There are 3 solutions:

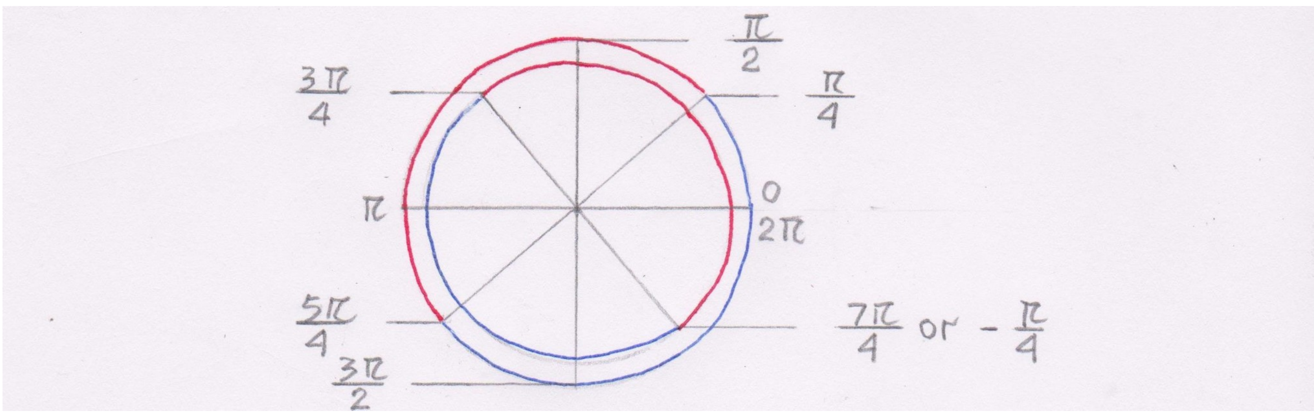
$x + \pi/4 = 0$, that gives $x = - \pi/4$ or $7\pi/4$ (co-terminal)
$x + \pi/4 = \pi$, that gives $x = \pi - \pi/4 = 3\pi/4$
$x + \pi/4 = 2\pi$, that gives $x = 2\pi - \pi/4 = 7\pi/4$

There are 2 end points at $(3\pi/4)$ and $(7\pi/4)$ and 2 arc lengths.
Find the sign status of g(x) by selecting the point $(\pi/2)$ as check point.
We have $g(\pi/2) = \sin \pi/2 + \cos \pi/2 = 1+ 0 = 1 > 0$. Then, $g(x) > 0$ inside the interval

(- Л/4, 3Л/4). Color it red and color the other arc length blue.

Figure the sign status of f(x) and g(x) on 2 concentric unit circles. By superimposing, we see that the combined solution set are the 2 open intervals **(Л/4, 3Л/4)** and **(5Л/4, 7Л/4)**.

Figure 1.



Exercises 2. Solve $F(x) = \sin^3 x + \cos^3 x \le 0$

Solution. Using the algebraic identity: $a^3 + b^3 = (a + b)(a^2 – ab + b^2)$, we get:

$F(x) = \sin^3 x + \cos^3 x = (\sin x + \cos x).(\sin^2 x – \sin x.\cos x + \cos^2 x)$
$F(x) = (\sin x + \cos x).(1 – \sin x.\cos x)$.

Transform again F(x) by using trig identities: (sin a + cos a) = √2sin (a + Л/4).
Finally we get:
F(x) = [√2sin(x + Л/4)].(1 – sin x.cos x) ≤ 0

a. Solve f(x) = sin (x + Л/4) = 0 . There are 3 solutions:

x + Л/4 = 0 that gives x = - Л/4 or x = 7Л/4 (co-terminal)
x + Л/4 = Л that gives x = Л – Л/4 = 3Л/4
x + Л/4 = 2Л that gives x = 2Л – Л/4 = 7Л/4.

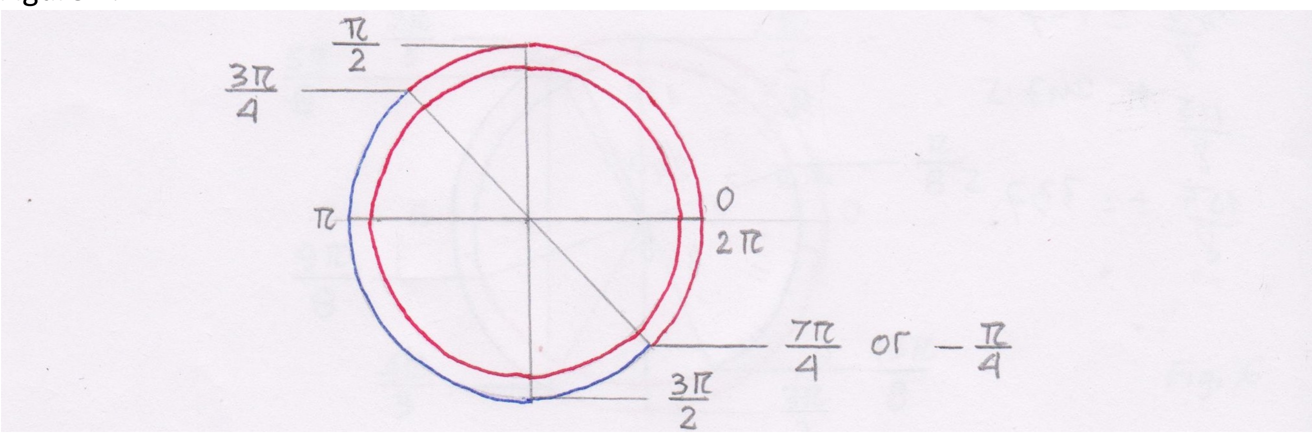
There are 2 end points at: x = 3Л/4 and x = 7Л/4 and 2 arc lengths.
To find the sign status of f(x) select the check point (Л/2).
We have: f(Л/2) = sin (Л/2 + Л/4) = sin (3Л/4) > 0. Therefore, f(x) is positive (> 0) inside the interval (-Л/4, 3Л/4). Color it red and color the other arc length blue.

b. Solve g(x) = 1 – sin x.cos x = 0

This function g(x) is always positive regardless of the values of x, since the product (sin x.cos x) is always less than 1.
Figure the sign status of f(x) and g(x) on 2 concentric unit circles.
By superimposing, we see that the combined solution set of F(x) ≤ 0 is the closed interval **[3Л/4, 7Л/4]**. See Figure 2.

Shortcut. Since g(x) is always positive, the sign status of F(x) is exactly the one of f(x). The solution set of F(x) ≤ 0 is the closed interval **[3Л/4, 7Л/4]**. Draw one unit circle is enough.

Figure 2.



Exercise 3. Solve F(x) = $2\sin^2 x - 3\sin x + 1 \geq 0$

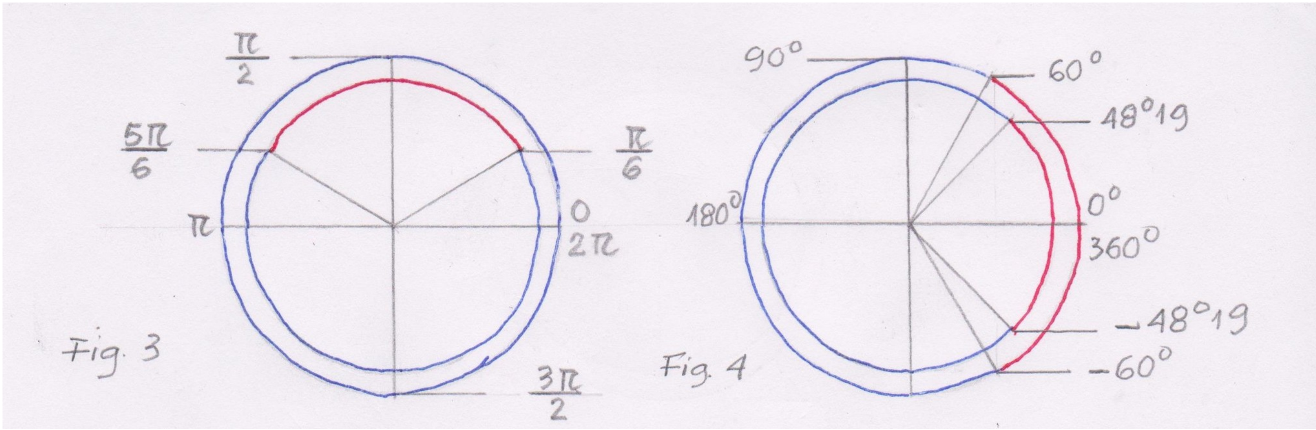
Solution. This is a quadratic equation in sin x. After factoring, we get:
F(x) = (sin x – 1)(2sin x – 1) ≥ 0.

a. Solve f(x) = sin x – 1 = 0. This function is always negative except when x = Л/2. Figure f(x) on a unit circle, and color the whole circle blue.

b. Solve g(x) = 2sin x – 1 = 0. This gives sin x = 1/2. There are 2 end points at (Л/6) and (5Л/6) and 2 arc lengths. To find the sign status of g(x), select point (Л/2) as check point.
We have g(Л/2) = 2sin Л/2 – 1 = 2 – 1 = 1 > 0. Then, g(x) is positive inside the arc length (Л/6, 5Л/6). Color it red and the rest blue. By superimposing, we see that the solution set is the closed interval **[5Л/6, 13Л/6].** See Figure 3.

Shortcut. Since f(x) is always negative, the sign status of F(x) is the opposite to the one of g(x). F(x) is positive (> 0) inside the closed interval [**5Л/6, 13Л/6**].

Figure 3 Figure 4



Exercise 4. Solve $F(x) = 6\cos^2 x - 7\cos x + 2 \leq 0$

Solution. It is a quadratic equation in cos x. After factoring we get:

$F(x) = 6\cos^2 x - 7\cos x + 2 = (2\cos x - 1)(3\cos x - 2) \leq 0$

a. Solve $f(x) = 2\cos x - 1 = 0$. This gives cos x = 0.5.

There are 2 end points at (60^o) and (- 60^o), and 2 arc lengths.
Select the point (10^o) as check point. We get: f(10) = 2cos (10) – 1 = 1.97 – 1 = 0.97 > 0.
Therefore, f(x) is positive inside the interval (- 60^o, 60^o). Color it red and color the rest blue.

b. Solve $g(x) = 3\cos x - 2 = 0$. This gives cos x = 2/3 = 0.67. There are 2 end points at (- 48^o19) and (48^o19), and 2 arc lengths.
Select point 0 as check point. We get: g(0) = 3 – 2 = 1 > 0. Therefore, g(x) > 0 inside the interval (-48^o19, 48^o19). Color it red and the rest blue.

By superimposing, we see that the solution set of F(x) ≤ 0 are the 2 closed intervals **[48^o19, 60^o]** and [**-60^o, -48^o19]**. See Figure 4.

Fast check by calculator. $F(x) = 6\cos^2 x - 7\cos x + 2 \leq 0$.

x = 55^o, this gives F(x) = 1.97 – 3.44 + 2 = - 0.53 < 0 (proved)
x = -55^o, this gives F(x) = 1.97 – 3.44 + 2 = - 0.53 < 0 (proved)
x = 180^o, this gives F(x) = 6 – 7 + 2 = 1 > 0 (proved)

NOTE. If the answers must be in radians, we convert the answers by the Rule of Conversion.

Exercise 5. Solve $F(x) = \cos 3x < 0$

Solution. Using trig identity: $\cos 3x = 4.\cos^3 x - 3\cos x$, we get
$F(x) = \cos x(4\cos^2 x - 3) = \cos x.(2\cos x - \sqrt{3}).(2\cos x + \sqrt{3}) < 0$

a. Solve $f(x) = \cos x = 0$. There are 2 end points at $(-\pi/2)$ and $(\pi/2)$. f(x) is positive inside interval $(-\pi/2, \pi/2)$. Color it red and the rest blue.

b. Solve $g(x) = 2\cos x - \sqrt{3} = 0$. This gives $\cos x = \sqrt{3}/2$ ---> $x = \pm (\pi/6)$.

There are 2 end points at $(-\pi/6)$ and $(\pi/6)$ and 2 arc lengths. Find the sign status of g(x) by selecting point (0) as check point. We have $g(0) = 2 - \sqrt{3} > 0$. Therefor, g(x) is positive inside arc length $(-\pi/6, \pi/6)$. Color it red and the rest blue.

c. Solve $h(x) = 2\cos x + \sqrt{3} = 0$. This gives $\cos x = -\sqrt{3}/2$ ---> $x = \pm (5\pi/6)$.

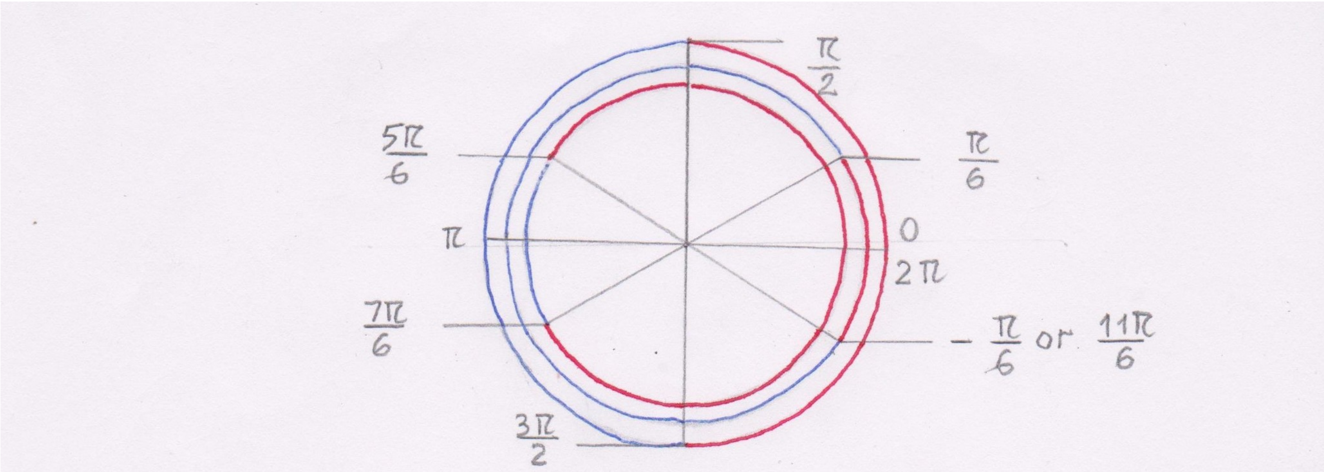
There are 2 end points at $(\pm 5\pi/6)$ and 2 arc lengths.
Select the point (π) as check point. We get: $h(x) = -2 + \sqrt{3} < 0$. Therefore, $h(x) < 0$ inside the interval $(5\pi/6, 7\pi/6)$. Color it blue and the rest red.
Figure the sign status of f(x), g(x) and h(x) on 3 concentric unit circles. By superimposing, we see that the combined solution set of $F(x) = \cos 3x < 0$ are the 3 open intervals (Figure 5):

$(\pi/6, \pi/2)$ where $f(x) > 0$, $g(x) < 0$, and $h(x) > 0$, and

$(5\pi/6, 7\pi/6)$ where f(x), g(x) and h(x) are all negative, and

$(3\pi/2, 11\pi/6)$ where $f(x) > 0$, $g(x) < 0$, and $h(x) > 0$

Figure 5



Fast check.

x = 0. This gives F(x) = cos 0 = 1 > 0 (Proved)
x = π. This gives F(x) = cos 3π = -1 < 0. (Proved)
x = π/3. This gives F(x) = cos π = - 1 < 0 (Proved)
x = 8π/6 = 4π/3. This gives F(x) = cos 12π/3 = cos 4π = 1 > 0 (Proved)

Exercise 6. Solve $F(x) = 6\sin^3x - 13\sin x + 9\sin x - 2 < 0$

Solution. Since a + b + c + d = 0, then, one real root is sin x = 1. After dividing by the factor (sin x – 1) we get:

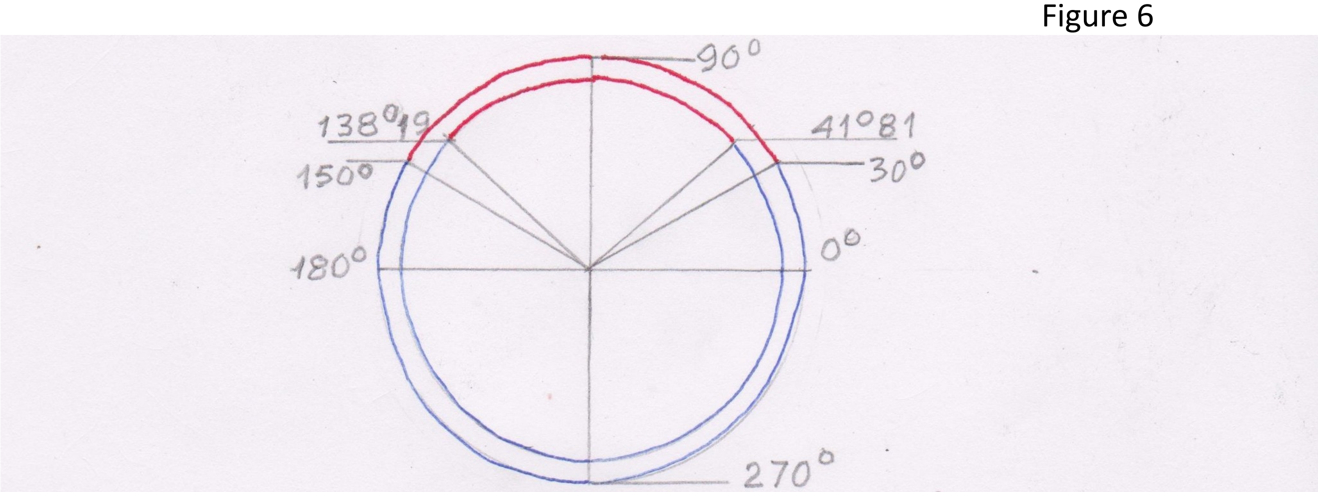
$F(x) = (\sin x - 1)(6\sin^2 x - 7\sin x + 2) < 0$. After factoring the trinomial, we get:
F(x) = f(x).g(x).h(x) = (sin x – 1).(2sin x -1).(3sin x – 2) < 0

a. f(x) = (sin x – 1) is always positive (except when x = π/2) regardless of x. Therefore, the sign status of F(x) is the sign of the product g(x).h(x).

b. Solve g(x) = 2sin x – 1 = 0 --→ sin x = 1/2. This gives 2 end points at (30^o) and (150^o) and 2 arc lengths. Find the sign status of g(x) by selecting check point (x = 90^o). We get:
g(90^o) = 2sin 90^o – 1 = 2 - 1 = 1 > 0. So, g(x) is positive inside the interval (30^o, 150^o) (Red). The rest is colored blue.

c. Solve h(x) = 3sin x – 2 --→ sin x = 2/3. This gives 2 end points at (41^o81) and (138^o19) and 2 arc lengths. Find the sign status of h(x) by selecting the check point (x = 90^o). We get:
h(90) = 3.sin 90 – 2 = 3 – 2 = 1 > 0. So, h(x) > 0, inside the interval (41^o81, 138^o19). Color it red and the rest blue

Figure 6



By superimposing, we see that the solution set are the 2 intervals:

(30^o, 41^o81) where g(x) > 0, and h(x) < 0 and
(138^o19, 150^o) where g(x) > 0, and h(x) < 0

Fast check by calculator. $g(x).h(x) = 6\sin^2 x - 7\sin x + 2 < 0$

x = 40^o. This gives: g(40).h(40) = 6(0.41) - 7(0.64) + 2 = 2.46 – 4.48 + 2 = - 0.02 < 0 (Proved)
x = 140^o. This gives: g(140).h(140) = 6(0.41) – 7(0.64) + 2 = - 0.02 < 0 (Proved)
x = 180^o. This gives: g(180).h(180) = 0 – 0 + 2 > 0 (OK)

REMARK.

1. To conveniently solve complex trig inequalities by Nghi H Nguyen method, students should remember these simple rules for end points and arc lengths:

- For f(x) = sin x and f(x) = cos x, or similar, there are 2 endpoints and 2 arc lengths for (0, 2Л)
- For f(x) = sin 2x & f(x) = cos 2x, or similar, there are 4 endpoints and 4 arc lengths for (0, 2Л)
- For f(x) = sin 3x & f(x) = cos 3x, or similar, there are 6 endpoints and 6 arc lengths for (0, 2Л)
- For f(x) = tan x, or similar, there are 1 endpoint, one discontinuation and 3 arc lengths for (0, Л)
- For f(x) = cot x, or similar, there are 1 endpoint, one discontinuation, and 3 arc lengths for (-Л/2, Л/2).

2. The Nghi Nguyen method is more convenient than the Sign Chart method because it can avoid the cut-offs at the 2 extremities of the sign chart. In the same process, it can show the periodic characteristic of trig functions when the origin point (0 or 2Л) is located inside an arc length.

(This article was authored by Nghi H Nguyen, Feb. 01, 2021)