SOLVING TRIG INEQUALITIES BY NGHI NGUYEN METHOD
Exercises (Part I)

(Authored by Nghi H Nguyen, Feb 01, 2021)

GENERALITIES.

To solve a complex trig inequality F(x), we must transform it into many basic trig functions (or similar) in the form:

$F(x) = f(x).g(x) \leq 0$ (or ≥ 0)   or $F(x) = f(x).g(x).h(x) \leq 0$ (or ≥ 0)

Transformation means include using common algebraic manipulations and trig identities. There are about 35 trig identities that are usually listed in trig books.

First, this method solves a basic trig function f(x) to find its end points that divide the unit circle into a few arc lengths. Next, it finds the sign status (+ or -) of the function f(x) in every arc length when the arc x rotates on the unit circle. To find the sign status of f(x), it uses the check point method, the same process as the number line checking point method.
Knowing the sign status of one arc length, we can quickly know the sign status of other arc lengths by using the rule of end points. Next, the method plots and color the sign status of f(x) on a unit circle, numbered in radians or degrees. It does the same way for other basic trig functions g(x) and h(x). By superimposing we can easily see the combined solution set.

Exercise 1. Solve $\sin^3 x + \cos^3 x > 0$ $(0, 2\pi)$

Solution. Using algebraic identity: $a^3 + b^3 = (a + b)(a^2 – ab + b^2)$, we get:
$F(x) = (\sin x + \cos x).(\sin^2 x – \sin x.\cos x + \cos^2 x)$
$F(x) = f(x).g(x) = (\sin x + \cos x)(1 – \sin x.\cos x) > 0$.

Since $g(x) = 1 – \sin x.\cos x$ is always positive regardless of x, therefore, the sign status of F(x) is the sign status of f(x).

Solve $f(x) = (\sin x + \cos x) = 0$. Use trig identity $(\sin a + \cos a) = \sqrt{2}\cos(a – \pi/4)$.

$f(x) = \sqrt{2}\cos(x – \pi/4) = 0$. This gives 2 solutions:
$x – \pi/4 = \pi/2$ --> that gives $x = \pi/2 + \pi/4 = 3\pi/4$
$x – \pi/4 = 3\pi/2 \rightarrow$ that gives $x = 3\pi/2 + \pi/4 = 7\pi/4$

There are 2 end points at $(3\pi/4)$ and $(7\pi/4)$, and 2 arc lengths. Find the sign status of f(x) by using point $(\pi/2)$ as check point.
We have: $f(\pi/2) = \cos(\pi/2) – \pi/4) = \cos \pi/4 > 0$.

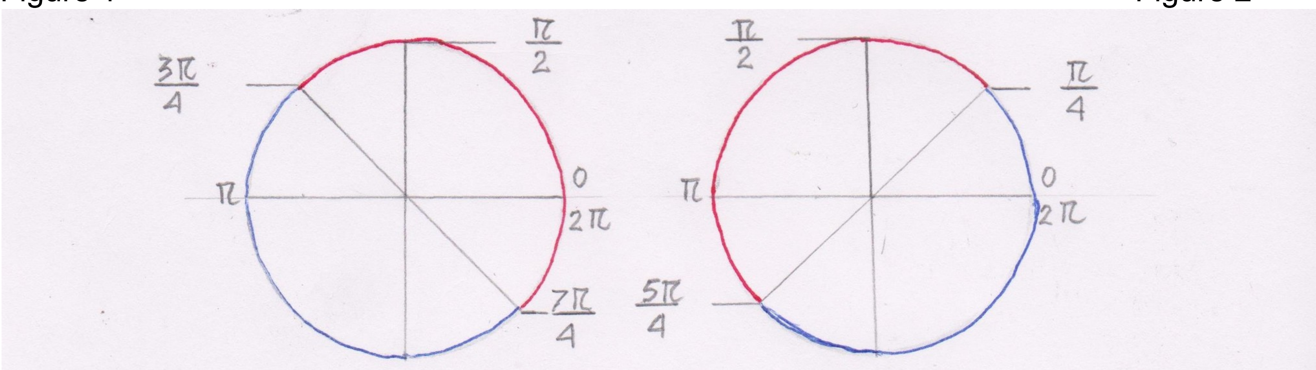
Therefore, f(x) is positive (> 0) in the interval ($-\pi/4$, $3\pi/4$). Color it red and the rest blue. On the unit circle, the solution set of F(x) > 0 (red) is the open interval **($-\pi/4$, $3\pi/4$).** See Figure 1.

Check $\sin^3 x + \cos^3 x > 0$

$x = \pi$. This gives: F(x) = 0 – 1 < 0 (proved)

$x = 0$. This give F(x) = 0 + 1 > 0 (proved)

Figure 1 Figure 2



Exercise 2. Solve $\sin^3 x - \cos^3 x \le 0$

Solution. Using algebraic identity: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, we get:

$F(x) = \sin^3 x - \cos^3 x = (\sin x - \cos x)(1 + \sin x.\cos x) \le 0$. (Period 2π)

Since the function $g(x) = (1 + \sin x.\cos x)$ is always positive regardless of x, the sign status of F(x) is the one of f(x).

Solve $f(x) = (\sin x - \cos x) = \sqrt{2}.\sin (x - \pi/4) = 0$. There are 3 solutions:

$(x - \pi/4) = 0$. this gives : $x = \pi/4$

$(x - \pi/4) = \pi$. This give : $x = \pi + \pi/4 = 5\pi/4$

$x - \pi/4 = 2\pi$. This gives: $x = 2\pi + \pi/4 = 9\pi/4$ or $\pi/4$

There are 2 end points at ($\pi/4$) and at ($5\pi/4$), and 2 arc lengths.
Find the sign status of f(x) by selecting point ($\pi/2$) as check point. We have $f(\pi/2) = \sin(\pi/2 - \pi/4) = \sin \pi/4 > 0$. Then, f(x) is positive (> 0) inside interval ($\pi/4$, $5\pi/5$). Color the arc length red and the rest blue.

On the unit circle, we see that the solution set of F(x) ≤ 0 (blue) is the closed interval **[$5\pi/4$, $9\pi/4$)**. See Figure 2.

Check. $\sin^3 x - \cos^3 x \le 0$

Select x = Л. This gives: F(x) = 0 - (-1) = 1 > 0. Proved
Select x = (3Л/2). This gives: F(x) = -1 + 0 = - 1 < 0. Proved.
Select x = 0. This gives F(x) = 0 – 1 < 0 . Proved.

Exercise 3. Solve sin 2x + cos 2x < 1

Solution. Change side of the inequality: F(x) =1 – cos 2x – sin 2x > 0

Using 2 trig identities (1 - cos 2a = $2\sin^2$ a) and (sin 2a = 2sina.cos a), we get:

F(x) = f(x).g(x) = 2sin x.(sin x – cos x) > 0

a. Solve sin x = 0. There are 2 end points at (0) and (Л). The function f(x) = sin x > 0 inside interval (0, Л). On the first unit circle, color it red and the other half circle blue.

b. Solve g(x) = sin x – cos x = √2.sin (x – Л/4) = 0. There are 3 solutions:

x – Л/4 = 0. This gives → x = Л/4
x – Л/4 = Л. This gives → x = 5Л/4
x – Л/4 = 2Л. This gives x = 2Л + Л/4 or x = Л/4

There are 2 endpoints at (Л/4) and (5Л/4), and 2 arc lengths. Find the sign status of g(x) by selecting point (Л/2) as check point. We have g(Л/2) = sin (Л/4) > 0. Therefor, g(x) > 0 inside the interval (Л/4, 5Л/4). Color it red and the rest blue.

By superimposing, we see that the solution set of F(x) > 0 are the 2 intervals
(Л/4, Л) and **(5Л/4, 2Л)**. See Figure 3

Figure 3

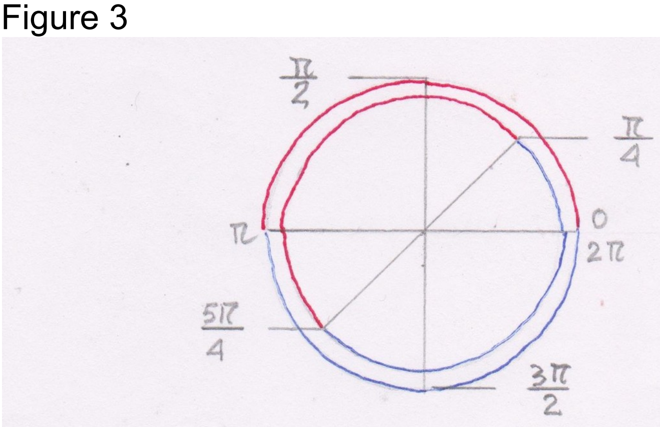
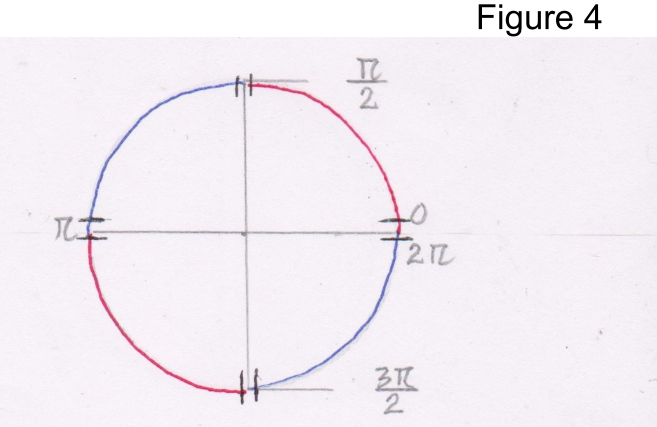


Figure 4



Check: $F(x) = 1 - \sin 2x - \cos 2x > 0$

Select $x = \pi/2$. This gives: $F(x) = 1 – 0 - (-1) = 2 > 0$ (proved)
Select $x = 3\pi/2$. This gives $F(x) = 1 – 0 - (-1) = 2 > 0$ (Proved)
Select $x = \pi/6$. This gives $F(x) = 1 – \sin \pi/3 – \cos \pi/3 = 1 - \sqrt{3}/2 – 1/2 < 0$ (Proved)

Exercise 4. Solve $\tan x + \cot x < 0$

Solution. Using definition of trig functions, transform the inequality.

$F(x) = \tan x + \cot x = \sin x/\cos x + \cos x/\sin x = (\sin^2 x + \cos^2 x)/(\sin x.\cos x) =$
$F(x) = 1/(\sin x.\cos x) = 2/\sin 2x$ ($F(x)$ undefined when $\sin 2x = 0$)

The sign status of $F(x)$ is the sign status of $f(x) = \sin 2x$.

Solve $f(x) = \sin 2x = 0$. There are 3 solutions:

$2x = 0$, that gives $x = 0 + k\pi$
$2x = \pi$, that gives $x = \pi/2 + k\pi$
$2x = 2\pi$, that gives $x = \pi + k\pi$
For $k = 1$, this gives $x = 3\pi/2$.

There are 4 discontinuity points at: (0), ($\pi/2$), (π), and ($3\pi/2$). There are 4 equal arc lengths. Find the sign status of $f(x) = \sin 2x$ by selecting $x = \pi/4$.
We get: $f(\pi/4) = \sin (\pi/2) = 1 > 0$.
Therefore, $f(x)$ is positive inside interval $(0, \pi/2)$. Color it red and color the 3 remainder arc lengths.
The solution set of $F(x) < 0$, same as of $f(x)$, are the 2 open intervals:
($\pi/2$, π) and **($3\pi/2$, 2π).** See Figure 4.

Exercise 5. Solve $\tan x + \cot x \geq 2$

Solution. From Exercise 4 we get: $(\tan x + \cot x) = 2/\sin 2x$.
Replace 2 by $(2\sin 2x)/\sin 2x$, we get:

$F(x) = (\tan x + \cot x) – 2 = (2/\sin 2x) – (2\sin 2x)/\sin 2x = [2(1 – \sin 2x)]/\sin 2x \geq 0$

The function $(1 – \sin 2x)$ is always positive regardless of x. Therefore, the sign status of $F(x)$ is the same as the one of $f(x) = \sin 2x$.
The solution set of $F(x) \geq 0$ are the 2 closed intervals:**[0, $\pi/2$]** and **[π, $3\pi/2$]**. Figure 4.
Note. There are 4 discontinuity points, but that doesn't affect the sign status of $f(x)$

Check. $\tan x + \cot x \geq 2$

Select $x = Л/3$. This give: $\tan Л/3 + \cot Л/3 = \sqrt{3} + \sqrt{3}/3 = 2.59 > 2$ (proved)
Select $x = 3Л/4$. This gives: $\tan (3Л/4) + \cot (3Л/4) = 1 + -1 = 0 < 2$. (Proved)
Select $x = 5Л/4$. This gives: $\tan (Л/4) + \cot (Л/4) = 1 + 1 = 2$ (Proved)
Select $x = 7Л/4$. This gives: $\tan (7Л/4) + \cot (7Л/4) = - 1 - 1 = - 2 < 2$ (Proved)

Example 6. Solve **$\sin 6x – \sin 4x < \sin 2x$** $(0, 2Л)$

Solution. Common period: $2Л$. Use these 2 trig identities to transform both sides:
$(\sin a – \sin b)$ and $\sin 2a = 2\sin a.\cos a$

$2\cos 5x.\sin x < 2\sin x.\cos x$
$2\cos 5x.\sin x – 2\sin x.\cos x < 0$
$2\sin x(\cos 5x – \cos x) < 0$
$2\sin x(- 2\sin 3x.\sin 2x) < 0$
$F(x) = f(x).g(x).h(x) = 4\sin x \sin 2x.\sin 3x > 0$ (side transposing)

Find the sign status of f(x), g(x), and h(x). Period $(0, 2Л)$

1. $f(x) = \sin x > 0$ when x is inside the arc length $(0, Л)$. Color it red. The rest of circle: blue.

2. Solve $g(x) = \sin 2x = 0 \rightarrow$ 2 solution arcs: $x = Л/2$ and $x = 3Л/2$

$g(x) = \sin 2x > 0$ (red) when x varies inside the 2 intervals $(0, Л/2)$ and $(Л, 3Л,2)$
$g(x) = \sin 2x < 0$ (blue) inside the 2 arc lengths: $(Л/2, Л)$ and $(3Л/2, 2$

3. Solve $g(x) = \sin 3x = 0$ to find the end points. There are 3 solution:

$3x = 0 \quad + 2kЛ$ $3x = Л \quad + 2kЛ$ $3x = 2Л+ 2kЛ$
$x = 2kЛ/3$ $x = Л/3 + 2kЛ/3$ $x = 2Л/3 + 2kЛ/3$

For k = 0, there are 3 end points:
$x = 0$ $x = Л/3$ $x = 2Л/3$
For k = 1, there are 3 end points:
$x = 2Л/3$ $x = Л$ $x = 4Л/3$
For k = 2, there are 3 end points
$x = 4Л/3$ $x = 5Л/3$ $x = 2Л$

There are totally 6 end points: (0), $(Л/3)$, $(2Л/3)$, $(Л)$, $(4Л/3)$, $(5Л/3)$, and 6 equal arc lengths.

Use check point method to find the sign status of of $g(x) = \sin 3x$ for $(0, 2\pi)$
Inside arc length $(0, \pi/3)$, select $x = \pi/6$ as check point. We get
$g(x) = \sin (3*\pi/6) = \sin (\pi/2) = 1 > 0$. Color this arc length **red**. See Figure 5
By the property of end points, we get the sign status of $g(x)$ inside the other arc lengths.

Arc length $(\pi/3, 2\pi/3)$ --> $g(x) < 0$ (blue).
Arc length $(2\pi/3, \pi) \rightarrow g(x) > 0$ (red)
Arc length $(\pi, 4\pi/3) \rightarrow g(x) < 0$ (blue)
Arc length $(4\pi/3, 5\pi/3) \rightarrow g(x) > 0$ (red)
Arc length $(5\pi/3, 2\pi) \rightarrow g(x) < 0$ (blue)

Figure 5.

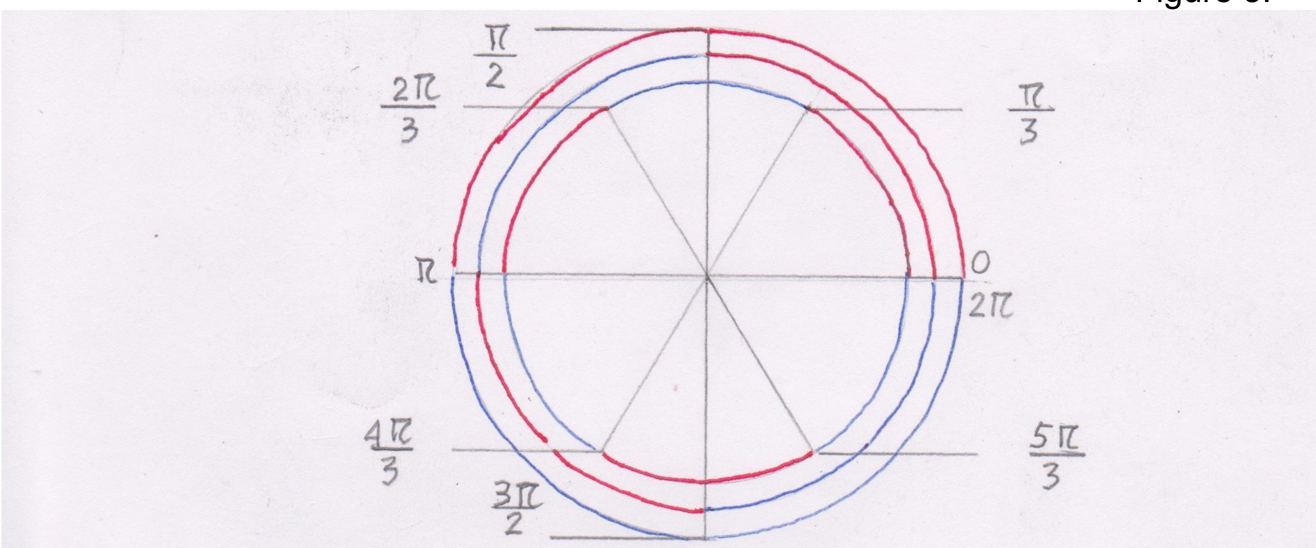


Figure separately the 3 sign status of $f(x)$, $g(x)$, and $h(x)$ on 3 concentric unit circles. By superimposing, we see that the combined solution set of $F(x) > 0$ are the 4 open intervals:

$(0, \pi/3)$ where the resulting sign is: $F(x) = (+)(+)(+) > 0$, and
$(\pi/2, 2\pi/3)$ where $\rightarrow F(x) = (+)(-)(-) > 0$, and
$(\pi, 4\pi/3)$ where $\rightarrow F(x) = (-)(+)(-) > 0$, and
$(3\pi/2, 5\pi/3)$ where $\rightarrow F(x) = (-)(-)(+) > 0$

Check. Check the original trig inequality: $\sin 6x - \sin 4x < \sin 2x$

$x = \pi/6 \rightarrow \sin (\pi) - \sin (\pi/3) = 0 - \sin (\pi/3) < \sin (\pi/3)$. Proved
$x = 5\pi/6 \rightarrow \sin (5\pi) - \sin (4\pi/3) = \sin (\pi) - (-\sin \pi/3) = 0 + \sin (\pi/3) > \sin (5\pi/3)$. Proved

REMARK

1. To conveniently solve complex trig inequalities by Nghi Nguyen method, students should remember these simple rules for end points and arc lengths:

- For f(x) = sin x and f(x) = cos x, or similar, there are 2 endpoints and 2 arc lengths for $(0, 2\pi)$
- For f(x) = sin 2x & f(x) = cos 2x, or similar, there are 4 endpoints and 4 arc lengths for $(0, 2\pi)$
- For f(x) = sin 3x & f(x) = cos 3x, or similar, there are 6 endpoints and 6 arc lengths for $(0, 2\pi)$
- For f(x) = tan x, or similar, there are 1 endpoint, one discontinuation and 3 arc lengths for $(0, \pi)$
- For f(x) = cot x, or similar, there are 1 endpoint, one discontinuation, and 3 arc lengths for $(-\pi/2, \pi/2)$.

2. The Nghi Nguyen method is more convenient than the Sign Chart method because it can avoid the cut-offs at the 2 extremities of the sign chart. In the same process, it can show the periodic characteristic of trig functions when the origin point (0 or 2π) is located inside an arc length.

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