SOLVING TRIG INEQUALITIES – SHOWCASE EXERCISES

(Authored by Nghi H Nguyen – Updated Feb. 16, 2021)

This article solves a few selected trig inequalities, using Nghi Nguyen method that has been recently posted on Google, Bing, Yahoo search.

Exercise 1. Solve $\cos 3x > 0$ (period 2π)

Solution. Use these trig identities to transform cos 3x:
$\cos (a + b) = \cos a.\cos b – \sin a.\sin b$
$\sin 2a = 2\sin.\cos a$
$1 – \cos 2a = 2\sin^2 a$. We get:

$F(x) = \cos (x+ 2x) = \cos x.\cos 2x – \sin x.\sin 2x > 0$
$F(x) = \cos x.\cos 2x – 2\sin^2 x.\cos x = \cos x(\cos 2x – 2\sin^2 x) > 0$
$F(x) = \cos x (\cos 2x – 1 + \cos 2x) = \cos x(2\cos 2x – 1) > 0$

1. Solve $f(x) = \cos x = 0$. The function $f(x) = \cos x$ is positive (> 0) inside the interval $(-\pi/2, \pi/2)$. Color it red and color the other half circle blue.

2. Solve $g(x) = 2\cos 2x – 1 = 0$. This gives 2 solutions:

$2\cos 2x = 1 \longrightarrow \cos 2x =1/2 = \cos (\pm \pi/3) + 2k\pi \longrightarrow x = (\pm \pi/6) + k\pi$.

There are 4 end points at $(\pi/6)$, $(5\pi/6)$, $(7\pi/6)$ and $(11\pi/6)$, and 4 arc lengths.
Find the sign status of g(x) by selecting point (0) as check point. We get :
$g(0) = 2\cos 0 – 1 = 2 – 1 = 1 > 0$. Therefor, $g(x) > 0$ inside the interval $(11\pi/6, 13\pi/6)$.
Color it red and color the other arc lengths.

Figure 1

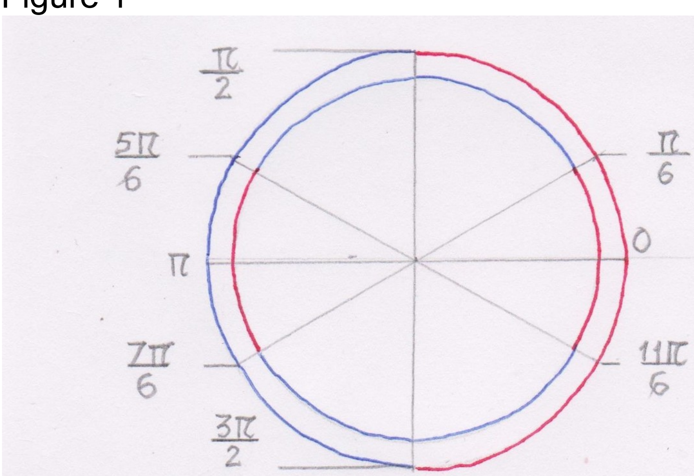
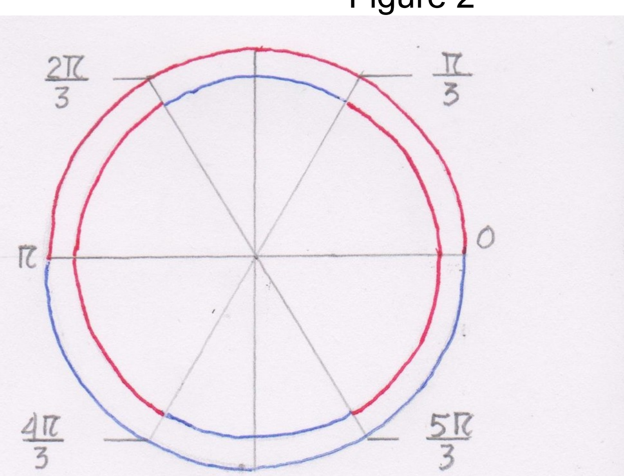


Figure 2



By superimposing, we see that the solution set of F(x) = cos 3x > 0 are the 3 open intervals: **(Л/2, 5Л/6), (7Л/6, 3Л/2)** and **(11Л/6, 13Л/6).** See Figure 1

Note.

1. We see that the sign chart has 2 separate extremities at x = 0 and x = 2Л. In this new method, the 2 extremities joint together and show the periodic character of trig functions. This is an advantage of the Nghi Nguyen Method.

2. This exercise aims to help students be familiar with the solving practice of complex trig inequalities by using the number unit circle. There is another method that directly solves the inequality cos 3x > 0 as a basic trig inequality.

Solve cos 3x = 0. That leads to 2 solutions;
3x = Л/2 + 2kЛ. This gives x = Л/6 + 2kЛ/3
3x = 3Л/2 + 2kЛ. This gives x = Л/2 + 2kЛ/3

For k = 0, k = 1, k = 2, there are 6 end points: Л/6, Л/2, 5Л/6, 7Л/6, 3Л/2, and 11Л/6
There are 6 equal arc lengths. The solution set for cos 3x > 0 are the 3 open intervals:

(Л/2, 5Л/6) and (7Л/6, 3Л/2) and (11Л/6, 13Л/6). Same answer as above.

Exercise 2. Solve sin 3x < 0

Solution. Use these trig identities to transform the inequality.

sin (a + b) = (sin a.cos b + sin b.cos a)
sin 2a = 2sin a.cos a
1 + cos 2a = $2\cos^2$ a.

F(x) = sin (x + 2x) = (sin x.cos 2x + sin 2x.cos x) = sin x.cos 2x + 2sin x.$\cos^2$ x < 0
F(x) = sin x(cos 2x + $2\cos^2$ x) = sin x(cos 2x + $2\cos^2$ x) > 0
F(x) = sin x(2cos 2x + 1) > 0

1. Solve f(x) = sin x. This function f(x) = sin x is positive (> 0) inside the interval (0, Л). Color it red and color the other half blue.

2. Solve g(x) = 2cos 2x + 1 = 0. This gives: cos 2x = - 1/2 ==> 2x = (± 2Л/3) + 2kЛ, and x = (± Л/3) + kЛ.

For k = 0 and k = 1, there are 4 end points at: (Л/3), (2Л/3), (4Л/3), and (5Л/4).

Find the sign status of $g(x) = 2\cos 2x + 1$ by selecting point (0) as check point. We get $g(0) = 2\cos(0) + 1 = 3 > 0$. Therefore, $g(x) > 0$ inside this interval $(11\pi/6, 13\pi/6)$.

The solution set for $f(x) = \sin 3x < 0$ are the 3 open intervals:

$(\pi/3, 2\pi/3)$, and **$(\pi, 4\pi/3)$**, and **$(5\pi/3, 2\pi)$.** See Figure 2

Note. We can directly solve the inequality ($\sin 3x < 0$) as a basic trig inequality. The answer will be the same.

Exercise 3. Solve $\sin 4x > 0$ (period 2π)

Solution. Use the trig identity ($\sin 2a = 2\sin a.\cos a$) to transform.

$F(x) = \sin 4x = 2\sin 2x.\cos 2x = 4\sin x.\cos x.\cos 2x > 0$

We have the form $F(x) = f(x).g(x).h(x) > 0$.

1. Solve $f(x) = \sin x = 0$. There are 2 end points at 0 and π. The function $f(x) = \sin x$ is positive inside the interval $(0, \pi)$. Color it red and the rest blue.

2. Solve $g(x) = \cos x = 0$. The function $g(x) = \cos x$ is positive (> 0) inside the interval $(-\pi/2, \pi/2)$. Color it red and the rest blue.

3. Solve $h(x) = \cos 2x = 0$. This give 2 solutions:

a. $2x = \pi/2 + 2k\pi \longrightarrow x = \pi/4 + k\pi$

b. $2x = 3\pi/2 + 2k\pi \longrightarrow x = 3\pi/4 + k\pi$

For k = 0 and k = 1, there are 4 end points at: $(\pi/4)$, $(3\pi/4)$, $(5\pi/4)$, and $(7\pi/4)$. There are 4 equal arc lengths.
Select point (0) as check point. We get $h(0) = \cos(0) = 1 > 0$. Therefore, $h(x)$ is positive (> 0), inside the interval $(-\pi/4, \pi/4)$. Color it red and color the other arc lengths.
By superimposing, the combined solution set of $F(x) > 0$ are the 4 open intervals:

$(0, \pi/4)$ where $f(x) > 0$, $g(x) > 0$ and $h(x) > 0$, and
$(\pi/2, 3\pi/4)$ where $f(x) > 0$, $g(x) < 0$ and $h(x) < 0$, and
$(\pi, 5\pi/4)$ where $f(x) < 0$, $g(x) < 0$ and $h(x) > 0$, and
$(3\pi/2, 7\pi/4)$ where $f(x) < 0$, $g(x) > 0$, and $h(x) < 0$. See Figure 3.

Figure 3

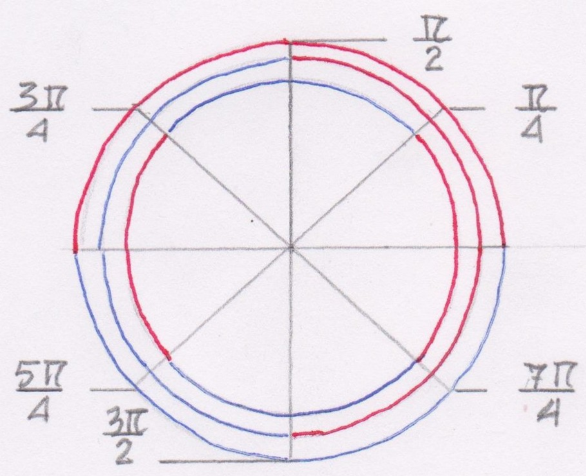
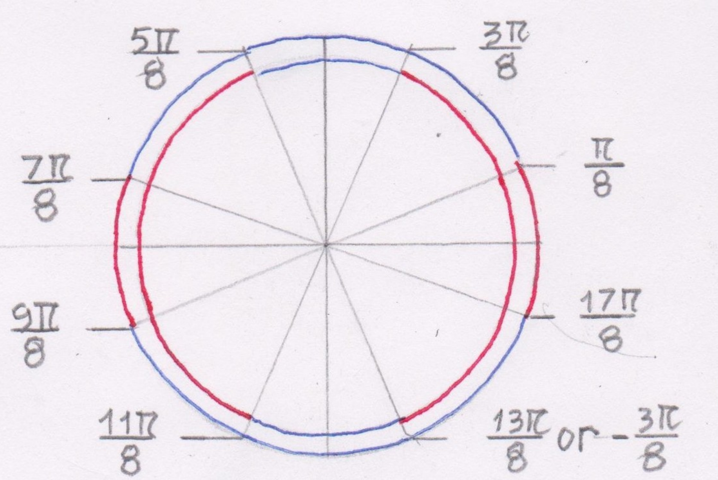


Figure 4



Exercise 4. Solve $\cos 4x > 0$

Solution. Use trig identity ($1 + \cos 2a = 2\cos^2 a$) to transform:

$F(x) = \cos 4x = 2\cos^2 2x – 1 = (\sqrt{2}\cos 2x – 1)(\sqrt{2}\cos 2x + 1) > 0$.

1. Solve $f(x) = \sqrt{2}\cos 2x – 1 = 0$ ---> $\cos 2x = \sqrt{2}/2 = \cos (\pm \pi/4)$ --→ $x = (\pm \pi/8) + k\pi$.

For $k = 0$, $k = 1$, there are 4 end points at: $(\pm\pi/8)$ and $(\pm 9\pi/8)$, and 4 arc lengths.

Select the check point ($x = 0$), we get: $f(0) = \sqrt{2} – 1 > 0$. Therefore, $f(x) > 0$ inside the interval $(- \pi/8, \pi/8)$. Color it red and color the other arc lengths.

2. Solve $g(x) = \sqrt{2}\cos 2x + 1 = 0$ --→ $2x = - \sqrt{2}/2 = \cos (\pm 3\pi/4) + 2k\pi$ --→
$x = (\pm 3\pi/8) + k\pi$.
For $k = 0$, and $k = 1$, there are 4 end points at $(\pm 3\pi/8)$ and $(\pm 11\pi/8)$ or at:
$(3\pi/8)$, $(5\pi/8)$, $(7\pi/8)$, and $(11\pi/8)$.
Select the point $(\pi/2)$ as check point. We get: $g(\pi/2) = \sqrt{2}\cos \pi + 1 = - \sqrt{2} + 1 < 0$.

Therefore, $g(x) < 0$ inside the interval $(3\pi/8, 5\pi/8)$. Color it blue and color the 3 other intervals.
By superimposing, the solution set of $F(x) = \cos 4x > 0$ are the 4 open intervals

$(3\pi/8, 5\pi/8)$ and **$(7\pi/8, 9\pi/8)$**, and **$(11\pi/8, 13\pi/8)$**, and **$(- \pi/8, \pi/8)$.** See Figure 4.

Note. We can directly solve the inequalities $\sin 4x > 0$ and $\cos 4x > 0$ as basic trig inequalities.

Exercise 5. Solve $\tan x + 2\tan^2 x < \cot x + 2$

Solution. Call $t = \tan x$, we have

$F(t) = t + 2t^2 – 1/t - 2 < 0$
$F(t) = t^2 + 2t^3 - 1 - 2t < 0$
$F(t) = t^2(1 + 2t) – (1 + 2t) < 0$
$F(t) = (1 + 2t)(t^2 – 1) = (1 + 2t)(t – 1)(t + 1) < 0$, or back to $\tan x = t$

$F(x) = (1 + 2\tan x)(\tan x – 1)(\tan x + 1) = f(x).(g(x).f(x) < 0$ (Common period 180º)

1. Solve $f(x) = 1 + 2\tan x = 0$. That gives $\tan x = - 1/2 = \tan (- 26º56) = \tan 153º43$. There is discontinuity point at x = 90º.

Find the sign status of f(x) by selecting point (x = 45º). We get: f(45) = 2(1) + 1 = 3 > 0

Therefor, f(x) > 0 inside the interval (0, 90º). Color it red and color the 2 other arc lengths..

2. Solve $g(x) = \tan x – 1 = 0$. This gives $\tan x = 1 = \tan 45º$. There is discontinuity point at x = 90º.
Select as checkpoint point (x = 20º). We get g(20) = 2(0.36) – 1 < 0. Therefore, g(x) is negative inside the interval (0, 45º). Color it blue.

3. Solve $h(x) = \tan x + 1 = 0$. This gives $\tan x = -1$ --→ x = 135º. Discontinuity at 90º

Select check point (20º). We get h(20) = 2(0.36) + 1 > 0. Therefore, h(x) > 0 inside the interval (0, 90º). Color it red.

By superimposing, the solution set of F(x) < 0 are the 3 open intervals:

(0, 45º), and **(90º, 135º)**, and **(153º43, 180º).** See Figure 5.

Fast check by calculator. $F(x) = (1 + 2\tan x)(\tan^2 x – 1) < 0$

x = 40º --→ F(40) = (1 + 0.84)(0.70 – 1) < 0. Proved.

x = 100º --→ F(100) = (1 – 5.67)(32.16 – 1) < 0. Proved

x = 170º --→ F(170) = (1 – 0.17)(0.03 – 1) < 0. Proved

Figure 5

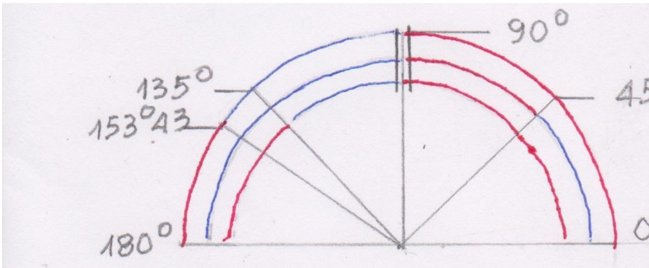
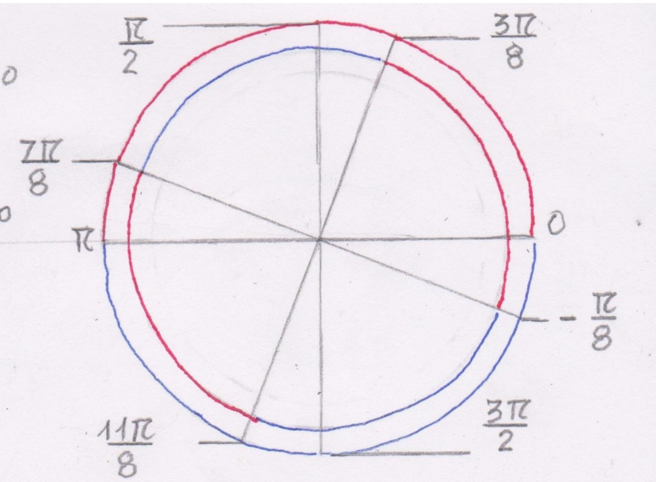


Figure 6



Exercise 6. Solve $\sin 3x - \sin x > \cos 3x - \cos x$

Solution. Use the trig identities (sin a – sin b) and (cos a – cos b) to transform the inequality:

$\sin 3x - \sin x > \cos 3x - \cos x$
$2\cos 2x.\sin x > - 2\sin 2x.\sin x$
$F(x) = 2\sin x.(\cos 2x + \sin 2x) > 0$

1. Solve $f(x) = \sin x = 0$. The function sin x is positive (> 0) inside the interval $(0, \pi)$. Color it red and the other half circle blue.

2. Solve $(\sin 2x + \cos 2x) = 0$. Use trig identity: $(\sin a + \cos a) = \sqrt{2}\cos (a + \pi/4)$.
We get: $(\sin 2x + \cos 2x) = \sqrt{2}.\cos (2x - \pi/4) = 0$. This gives 2 solutions:

a. $2x - \pi/4 = \pi/2 + 2k\pi \longrightarrow 2x = 3\pi/4 + 2k\pi \longrightarrow x = 3\pi/8 + k\pi$
b. $2x - \pi/4 = 3\pi/2 + 2k\pi \longrightarrow 2x = 7\pi/4 + 2k\pi \longrightarrow x = 7\pi/8 + k\pi$

For k = 0, and k = 1, there are 4 end points at: $(3\pi/8)$, $(7\pi/8)$, $(11\pi/8)$, and $(15\pi/8)$. There are 4 arc lengths. Find the sign status of g(x) by selecting the check point (0). We get $g(0) = (\sin 0 + \cos 0) = 1 > 0$. Therefor, g(x) is positive (> 0) inside the interval $(- \pi/8, 3\pi/8)$. Color it red and color the 3 other arc lengths.

By superimposing, we see that the solution set of F(x) > 0 are the 3 open intervals; **$(0, \pi/8)$** and **$(7\pi/8, \pi)$** and **$(11\pi/8, 15\pi/8)$**. See Figure 6.

Check. $F(x) = \sin x.\cos (2x – \pi/4) > 0$
$x = \pi/4$ --→ $F(\pi/4) = \sin \pi/4.\cos (\pi/2 – \pi/4) = 1/2 > 0$. Proved
$x = \pi/2$ --→ $F(\pi/2) = \sin \pi/2.\cos (3\pi/4) = (+)(-) < 0$. Proved
$x = 3\pi/2$ --→ $F(3\pi/2) = (-1)\cos (3\pi -\pi/4) = (-1)\cos (3\pi/4) = (-)(-) > 0$. Proved

Exrcise 7. Solve $\tan^3 x + \tan^2 x > 3\tan x + 3$

Solution. Put ($\tan x + 1$) in common factor.

$\tan^2 x(\tan x + 1) > 3(\tan x + 1)$. We get in standard form:
$F(x) = \tan^2 x.(\tan x + 1) – 3(\tan x + 1) > 0$
$F(x) = (\tan x + 1)(\tan^2 x – 3) = (\tan x + 1)(\tan x - \sqrt{3})(\tan x + \sqrt{3}) > 0$

1. Solve $f(x) = \tan x + 1 = 0$. This gives $\tan x = -1$ --→ $x = 3\pi/4$
There is discontinuity at $x = \pi/2$. Find the sign status of f(x) by selecting the check point $x = \pi/4$. We get: $f(\pi/4) = \tan \pi/4 + 1 > 0$. Then, f(x) is positive inside the interval $(0, \pi/2)$. Color it red and color the 2 other arc lengths.

2. Solve $g(x) = \tan x - \sqrt{3}$. This gives: $\tan x = \sqrt{3} = \tan \pi/3$. Discontinuity at $x = \pi/2$. Select check point ($x = \pi/4$). We have: $g(\pi/4) = \tan \pi/4 - \sqrt{3} < 0$. Then, $g(x) < 0$ inside the arc length $(0, \pi/2)$. Color it blue.

3. Solve $h(x) = \tan x + \sqrt{3}$. This gives: $\tan x = \sqrt{3} = \tan (2\pi/3)$. Discontinuity at $\pi/2$. Select ($x = \pi/4$) as check point. We get: $h(x) = \tan \pi/4 + \sqrt{3} > 0$. Then, $h(x) > 0$ inside the arc length $(0, \pi/2)$. Color it red

The solution set of $F(x) > 0$ are the 2 open intervals
$(\pi/3, \pi/2)$ and **$(2\pi/3, 3\pi/4)$.**

Fast check by calculator.
$x = 80^o$ --→ $F(80) = (5.67 + 1)(5.67 - \sqrt{3})(5.67 + \sqrt{3}) > 0$ Proved
$x = 130^o$ --→ $F(130) = (-1.19 + 1)(- 1/19 - \sqrt{3})(-1.19 + \sqrt{3}) > 0$. Proved

Exercise 8. Solve $F(x) = (\sin 2x)/(1 - \sqrt{2}\cos x) > 0$.

Solution. We have the case $F(x) = f(x)/g(x)$.

1. Solve $f(x) = \sin 2x = 0$. This gives 3 solutions:

$\sin 2x = 0 + 2k\pi$. This gives $x = 0 + k\pi$
$\sin 2x = \pi + 2k\pi$. This gives $x = \pi/2 + k\pi$
$\sin 2x = 2\pi + 2k\pi$. This gives $x = \pi + k\pi$

Figure 7

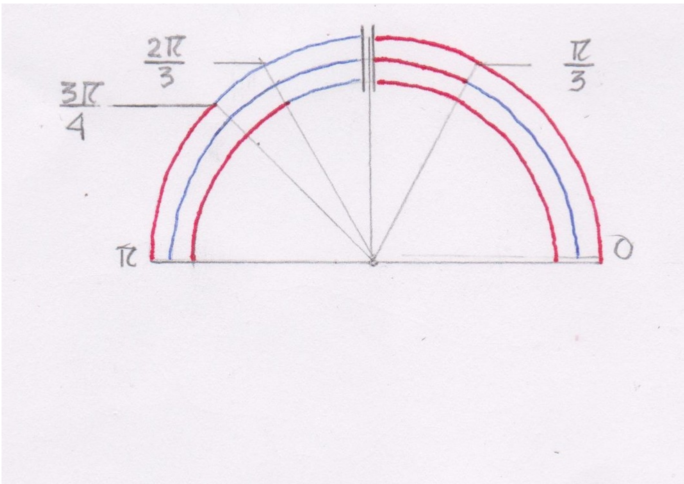
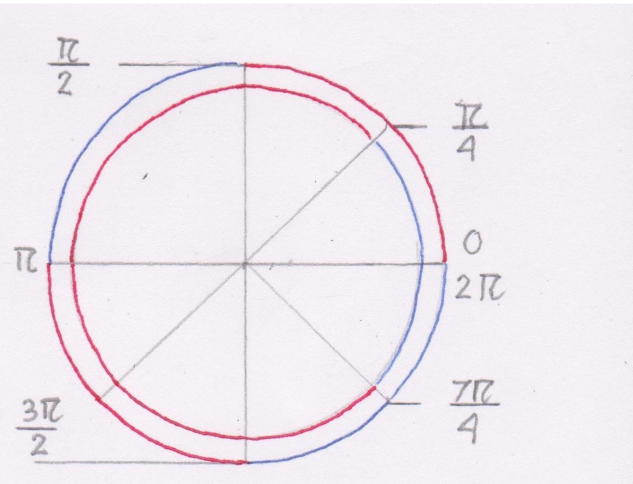


Figure 8



For k = 0 and k = 1, there are 4 end points at: 0, $\pi/2$, π, and $3\pi/2$. There are 4 arc lengths. $f(x) = \sin 2x$ is positive inside the arc length $(0, \pi/2)$. Color it red and color the 3 other arc lengths.

2. Solve $g(x) = 1 - \sqrt{2}\cos x$. This gives $\cos x = \sqrt{2}/2 = \cos(\pm\pi/4)$. There are 2 arc lengths. Select point (0) as check point. We get: $g(0) = 1 - \sqrt{2} < 0$. Therefor, g(x) is negative (< 0) inside the interval $(-\pi/4, \pi/4)$. Color it blue.

The solution set of $F(x) > 0$ are the 3 open intervals:

$(\pi/4, \pi/2)$, and **$(\pi, 3\pi/2)$** and **$(7\pi/4, 2\pi)$**

REMARK

1. The sign chart method is inconvenient since all values of x of all operations are progressively placed on the first line only. That makes the chart **stuffy**, complicated, and **confusing**. In the Nghi Nguyen method, values of x go each time with only one basic function.

2. The sign chart has a disadvantage since its 2 extremities are **separate**. In the new method, on the unit circle, these 2 extremities joint together and show the periodic character of trig functions whenever the origin (0 or 2π) is located inside an arc length.

(This article was authored by Nghi H Nguyen, Updated Feb.16, 2021)