

SOLVING TRIGONOMETRIC INEQUALITIES (CONCEPT, METHODS, AND STEPS)

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DEFINITION.

A trig inequality is an inequality in standard form: $R(x) > 0$ (or < 0) that contains one or a few trig functions of the variable arc x . Solving the inequality $R(x)$ means finding all the values of the variable arc x whose trig functions make the inequality $R(x)$ true.

All these values of x constitute the **solution set** of the trig inequality $R(x)$.

Solution sets of trig inequalities are expressed in **intervals**.

Examples of trig inequalities:

$$\sin(x + 30^\circ) < 0.75$$

$$\tan x + \cot x > 2$$

$$\sin(2x + \pi/3) < 0.5$$

$$\sin x + \sin 2x < -\sin 3x$$

$$\cos 2x + 3\sin x > 1.5$$

$$2 \tan x + \cot x > 3$$

Example of solution sets of trig inequalities in the form of intervals:

$(\pi/4, 2\pi/3)$; $[0, 2\pi]$; $[-\pi/2, \pi/2]$; $(20^\circ, 80^\circ)$; $(30^\circ, 120^\circ)$

THE TRIG UNIT CIRCLE

It is a circle with radius $R = 1$ unit, with an origin O . The variable arc AM that rotates counterclockwise on the trig unit circle defines 4 common trig functions of the arc x .

When an arc AM varies on the trig unit circle:

The horizontal axis OAx defines the trig function $f(x) = \cos x$.

The vertical axis OBy defines the trig function $f(x) = \sin x$.

The vertical axis At defines the trig function $f(x) = \tan x$.

The horizontal axis Bu defines the trig function $f(x) = \cot x$.

The trig unit circle will be used as proof in solving basic trig equations and basic trig inequalities.

COMMON PERIOD OF THE TRIG INEQUALITY

The common period of a trig inequality is the least multiple of all periods of the trig functions presented in the inequality. Examples:

The trig inequality: $\sin x + \sin 2x + \cos x/2 < 1$ has 4π as common period.

The trig inequality: $\tan 2x + \sin x - \cos 2x > 2$ has 2π as common period.

The trig inequality: $\tan x + \cos x/2 < 3$ has 4π as common period.

Unless specified, a trig inequality must be solved, at least, within one whole common period.

BASIC TRIG INEQUALITIES.

There are 4 main common types of basic trig inequalities:

$$\begin{array}{lll} \sin x < a \text{ (or } > a) & \cos x < a \text{ (or } > a) & a \text{ is a given number} \\ \tan x < a \text{ (or } > a) & \cot x < a \text{ (or } > a) & \end{array}$$

Solving basic trig inequalities proceeds by using trig conversion tables (or calculators), then by considering the various positions of the variable arc x that rotates on the trig circle.

Example 1. Solve the inequality: $\sin x > 0.709$

Solution. The solution set is given by both trig table and trig unit circle. On the trig unit circle, $\sin x > 0.709$ when the arc x varies between $\pi/4$ and $3\pi/4$:

$$\begin{array}{ll} \pi/4 < x < 3\pi/4 & \text{Answer within period } 2\pi \\ \pi/4 + 2k\pi < x < 3\pi/4 + 2k\pi & \text{Extended answers} \end{array}$$

Example 2. Solve: $\tan x < 0.414$

Solution. The solution set is given by the unit circle and calculator. On the trig unit circle, $\tan x > 0.414$ when the arc x varies between the values $-\pi/2$ (or $3\pi/2$) and $\pi/8$.

$$\begin{array}{ll} -\pi/2 < x < \pi/8 & \text{Answer within period } \pi \\ -\pi/2 + k\pi < x < \pi/8 + k\pi & \text{Extended answers} \end{array}$$

Example 3. Solve: $\cos(2x + \pi/4) < 0.5$ within period 2π

Solution. Solution set given by unit circle and calculator:

$$\begin{array}{ll} \pi/3 < 2x + \pi/4 < 5\pi/3 & \\ \pi/12 < 2x < 17\pi/12 & \\ \pi/24 < x < 17\pi/24 & \text{Answer} \end{array}$$

Example 4. Solve: $\cot(2x - \pi/6) < -0.578$ (within period π)

Solution. Solution set given by trig circle and calculator:

$$\begin{array}{ll} 2\pi/3 < x - \pi/6 < \pi & \\ 5\pi/6 < 2x < 7\pi/6 & \\ 5\pi/12 < x < 7\pi/12 & \text{Answer} \end{array}$$

To fully know how to solve basic trig inequalities, and similar, see book titled: "Trigonometry: Solving trigonometric equations and inequalities" (Amazon e-book 2010)

SOLVING CONCEPT

To solve a trig inequality, transform it into one or many trig inequalities. Solving trig inequalities finally results in solving basic trig inequalities.

To transform a trig inequality into basic ones, students can use common algebraic transformations (common factor, polynomial identities...), definitions and properties of trig functions, and trig identities, the most needed. There are about 31 trig identities, among them the last 14 identities (from # 19 to # 31) are called **transformation identities**, since they are necessary tools to transform trig inequalities (or trig equations) into basic ones. See book mentioned above.

Example 5. Transform the inequality $\sin x + \cos x < 0$ into a product.

Solution.
$$\begin{aligned}\sin x + \cos x &= \sin x + \sin \left(\frac{\pi}{2} - x\right) = \\ &= 2\sin \frac{\pi}{4} \cdot \sin \left(x + \frac{\pi}{4}\right) < 0 \quad \text{Use Sum into Product Identity, \#28}\end{aligned}$$

Example 6. Transform the inequality $\sin 2x - \sin x > 0$ into a product

Solution.
$$\begin{aligned}\sin 2x - \sin x &= 2\sin x \cdot \cos x - \sin x \\ &= \sin x (2\cos x - 1) > 0 \quad \text{Trig identity \& common factor}\end{aligned}$$

Example 7. Transform $(\cos 2x < 1 + \sin x)$ into a product.

Solution.
$$\begin{aligned}\cos 2x - 1 - \sin x &< 0 \\ 1 - 2\sin^2 x - 1 - \sin x &< 0 \quad (\text{Replace } \cos 2x \text{ by } 1 - 2\sin^2 x) \\ -\sin x(2\sin x + 1) &< 0\end{aligned}$$

Important Note. The transformation process for the inequality $R(x) > 0$ (or < 0) is exactly the same as the transformation process of the equation $R(x) = 0$. Solving the trig inequality $R(x)$ requires first to solve the equation $R(x) = 0$ to get all of its real roots.

STEPS IN SOLVING TRIG INEQUALITIES

There are 4 steps in solving trig inequalities.

Step 1. Transform the given trig inequality into standard form $R(x) > 0$ (or < 0).

Example. The inequality $(\cos 2x < 2 + 3\sin x)$ will be transformed into standard form:
 $R(x) = \cos 2x - 3\sin x - 2 < 0$

Example. The inequality $(\sin x + \sin 2x > -\sin 3x)$ will be transformed into standard form $R(x) = \sin x + \sin 2x + \sin 3x > 0$.

Step 2. Find the common period . The common period must be the least multiple of the periods of all trig functions presented in the inequality. The complete solution set must, at least, cover one whole common period.

Example. The trig inequality $R(x) = \cos 2x - 3\sin x - 2 < 0$ has 2π as common period

Example. The trig inequality $R(x) = \sin x - \cos x/2 - 0.5 > 0$ has 4π as common period.

Example. The trig inequality $R(x) = \tan x + 2 \cos x + \sin 2x < 2$ has 2π as common period.

Step 3. Solve the trig equation $R(x) = 0$

If $R(x)$ contains only one trig function, solve it as a basic trig equation.

If $R(x)$ contains 2 or more trig functions, there are 2 methods, described below, to solve it.

- a. **METHOD 1.** Transform $R(x)$ into a product of many basic trig equations. Next, solve these basic trig equations separately to get all values of x that will be used in Step 4.

Example 8. Solve: $\cos x + \cos 2x > -\cos 3x$ ($0 < x < 2\pi$)

Solution. Step 1. Standard form: $R(x) = \cos x + \cos 2x + \cos 3x > 0$

Step 2. Common period: 2π

Step 3. Solve $R(x) = 0$. Transform it into a product using Sum to Product Identity:

$$R(x) = \cos x + \cos 2x + \cos 3x = \cos 2x (1 + 2\cos x) = 0.$$

Next, solve the 2 basic trig equations $f(x) = \cos 2x = 0$ and $g(x) = (1 + 2\cos x) = 0$ to get all values of x within the period 2π . These values of x will be used in Step 4.

Example 9. Solve $\sin x + \sin 2x < -\sin 3x$ ($0 < x < 2\pi$)

Solution. Step 1: $\sin x + \sin 2x + \sin 3x < 0$

Step 2: Common period 2π .

Step 3. Solve $R(x) = 0$. Transform it into a product using trig identity:

$$R(x) = \sin x + \sin 2x + \sin 3x = \sin 2x (2\cos x + 1) = 0$$

Next, solve the 2 basic trig equations $f(x) = \sin 2x = 0$ and $g(x) = 2\cos x + 1 = 0$. The found values of x will be used in Step 4.

b. **METHOD 2.** This method transforms a trig inequality with 2 or more trig functions into a trig inequality having only one trig function (called t) as variable. Next, solve for t from this trig equation, as a basic trig equation. Then, solve for x from these values of t . The common trig functions to be chosen as function variable are: $\sin x = t$; $\cos x = t$, $\cos 2x = t$; $\tan x = t$; and $\tan x/2 = t$.

Example 10. Solve: $3\sin^2 x = \sin 2x + \cos^2 x$

Solution. Divide both sides by $\cos^2 x$ ($\cos x$ not equal 0; x not equals $\pi/2$). Let $\tan x = t$.

$$3t^2 - 2t - 1 = 0 \quad \text{This is a quadratic equation having 2 real roots: 1 and } -1/3$$

Next, solve the 2 basic trig equations: $\tan x = t = 1$ and $\tan x = t = -1/3$

Example 11. Solve $\tan x + 2 \tan^2 x = \cot x + 2$ ($0 < x < \pi$)

Solution. Let $\tan x = t$. (with t not equal $\pi/2$; t not equal π)

$t + 2t^2 = 1/t + 2$	Let $\tan x = t$
$t^2(1 + 2t) = 1 + 2t$	Multiply both sides by t
$(2t + 1)(t^2 - 1) = 0$	Factor out $(2t + 1)$

Next, solve the 3 basic trig equations: $(2\tan x + 1) = 0$; $\tan x = -1$; $\tan x = 1$.

Step 4. Solve the trig inequality $R(x) < 0$ (or > 0). Then express the solution set in the form of intervals.

Based on the found values of x from Step 3, solve, **algebraically**, the trig inequality $R(x) < 0$ (or > 0) by separately solving each basic trig inequality $f(x)$, $g(x)$.., and then by setting up a **sign sable** (sign chart).

Example 12. Solve: $\sin x + \sin 3x < -\sin 2x$

Solution. Step 1. Standard form: $R(x) = \sin x + \sin 2x + \sin 3x < 0$

Step 2. Common period 2π

Step 3. Solve $R(x) = 0$. Transform $R(x)$ into a product, using Trig Identity # 28;

$$R(x) = 2\sin 2x (2\cos x + 1) = 0 \quad (0 < x < 2\pi)$$

Next, Solve the basic trig equation: $f(x) = \sin 2x = 0$. The solution arcs x are: $0, \pi/2, \pi, 3\pi/2, 2\pi$. Using the trig unit circle as proof, determine the variation of $f(x)$ from 0 to 2π with (+) and (-) values:

x	0		Pi/2		Pi		3Pi/2		2Pi
f(x)	0	+	0	-	0	+	0	-	0

Solve $g(x) = 2\cos x + 1 = 0$. The solution arcs are: $2\pi/3, 4\pi/3$. Determine the variation of $g(x)$ from 0 to 2π with (+) and (-) values:

x	0		$2\pi/3$		$4\pi/3$		2π
g(x)		+	0	-	0	+	

Step 4. Solve the inequality $R(x) < 0$ by the **algebraic method**.

First, create a **sign table**, in which the top line figures all values of x obtained from Step 3, and progressively varying from 0 to 2π . These x values create various **intervals** between them.

Next, determine the variations of $f(x)$ and $g(x)$ on the second and third line of the table, by putting + or - sign inside each corresponding interval.

The last line of the table figures the variation of $R(x)$ that has the **resulting sign** of the product $f(x) \cdot g(x)$. In this Example 12, all the intervals where $R(x)$ is negative (< 0) constitute the solution set of the trig inequality $R(x) < 0$.

THE SIGN TABLE (SIGN CHART)

x	:	0		$\pi/2$		$2\pi/3$		π		$4\pi/3$		$3\pi/2$		2π
f(x)	:	0	+	0	-		-	0	+		+	0	-	0
g(x)	:		+		+	0	-		-	0	+		+	
R(x)	:	0	+	0	-	0	+	0	-	0	+	0	-	0
<div style="display: flex; justify-content: space-around; width: 100%;"> Yes Yes Yes </div>														

Solution set of the trig inequality $R(x) < 0$: $(\pi/2, 2\pi/3)$ and $(\pi, 4\pi/3)$ and $(3\pi/2, 2\pi)$

NOTE 1. The approach to determine the variations of $f(x)$ and $g(x)$ is exactly the same approach in solving basic trig inequalities, basing on the positions of the variable arc x that rotates on the trig unit circle.

NOTE 2. THE GRAPHING METHOD

Solving trig inequalities is a tricky work that often leads to errors/mistakes. After solving trig inequalities by the algebraic method, you can check the answers by graphing the trig function $R(x)$ by graphing calculators.

You can also use graphing calculators to directly solve the trig inequality $R(x) < 0$ (or > 0). This method, if allowed, is fast, accurate and convenient. To know how to proceed, read the last chapter of the book mentioned above (Amazon e-book 2010).

EXAMPLES ON SOLVING TRIG INEQUALITIES

Example 13. Solve: $(2\cos x - 1)/(2\cos x + 1) < 0$ $(0 < x < 2\pi)$

Solution.

Step 1: The function $F(x) = f(x)/g(x) = (2\cos x - 1)/(2\cos x + 1) < 0$ is undefined when $x = 2\pi/3$ and $x = 4\pi/3$.

Step 2. Common period 2π .

Step 3. Solve $f(x) = 2\cos x - 1 = 0 \rightarrow \cos x = 1/2 \rightarrow x = \pi/3 ; x = 5\pi/3$

Solve $g(x) = 2\cos x + 1 = 0 \rightarrow \cos x = -1/2 \rightarrow x = 2\pi/3 ; x = 4\pi/3$

Step 4. Solve $F(x) < 0$, algebraically, by setting up the sign chart (sign table)

x		0	$\pi/3$		$2\pi/3$		$4\pi/3$		$5\pi/3$		2π
f(x)		+	0	-		-		-	0	+	

g(x)		+		+	0	-	0	+		+	

F(x)		No	I	Yes	II	No	II	Yes	I	No	

Answer or solution set: $(\pi/3, 2\pi/3) ; (4\pi/3, 5\pi/3)$.

Example 14. Solve: $\tan x + \cot x < -4$ $(0 < x < \pi)$

Solution.

Step 1. Transform the inequality into standard form;

$$\sin x/\cos x + \cos x/\sin x + 4 = 2/\sin 2x + 4 < 0$$

$$F(x) = 2(2\sin 2x + 1)/\sin 2x < 0 \quad \text{undefined when } x = k\pi/2 \text{ and } x = k\pi$$

Step 2. The common period of $F(x)$ is π since the period of $\sin 2x$ is π .

Step 3. First, solve $f(x) = 2\sin 2x + 1 = 0$

$$\sin 2x = -1/2 \rightarrow x = 7\pi/2 ; x = 11\pi/12$$

Next, solve $g(x) = \sin 2x = 0 \rightarrow \sin 2x = 0 \rightarrow x = 0 ; x = \pi/2$

Step 4. Solve $F(x) < 0$, algebraically, by setting up the sign chart (sign table):

x	I	0		$\pi/2$		$7\pi/2$		$11\pi/2$		π
$f(x)$	I		+		+	0	-	0	+	
$g(x)$	I	0	+	0	-		-		-	0
$F(x)$	I	II	No	II	Yes	I	No	I	Yes	II

Answer or solution set: $(\pi/2 , 7\pi/2) ; (11\pi/12 , \pi)$

(This article was written by Nghi H. Nguyen, the co-author of the new Diagonal Sum Method for solving quadratic equations)