SOLVING TRIGONOMETRIC EQUATIONS – CONCEPT & METHODS

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DEFINITION.

A trig equation is an equation containing one or many trig functions of the **variable arc x** that rotates counter clockwise on the trig unit circle. Solving for **x** means finding the values of the trig arcs x whose trig functions make the trig equation true.

Example of trig equations:

tan (x + Pi/3) = 1.5	sin (2x + Pi/4) = 0.5	sin x + sin 2x = 0.75
cos x + sin 2x = 1.2	tan x + cot x = 1.732	2sin 2x + cos x = 1

Answers, or values of the solution arcs, are expressed in degrees or radians.

Examples:	x = 30 degree;	x = 43.72 degree	x = 360 deg.
	x = Pi/3	x = 2Pi/3	x = 2Pi

THE TRIG UNIT CIRCLE.

It is a circle with radius R = 1 with an origin O. The unit circle defines the main trig functions of the variable arcs x that rotates counterclockwise on it.

When the variable arc AM, with value x (in radians or degree) varies on the trig unit circle:

The horizontal axis OAx defines the function **f(x) = cos x.**

The vertical axis OBy defines the function **f(x) = sin x**.

The vertical axis AT defines the function **f(x) = tan x**

The horizontal axis BU defines the function **f(x) = cot x**.

The trig unit circle will be used as proof for solving basic trig equations & basic trig inequalities.

THE PERIODIC PROPERTY OF TRIG FUNCTIONS.

All trig functions are periodic meaning they come back to the same values after the trig arc x rotates one period on the trig unit circle. Examples:

The trig function f(x) = sin x has 2Pi as period
The trig function f(x) = tan x has Pi as period
The trig function f(x) = sin 2x has Pi as period

The trig function f(x) = cos (x/2) has 4Pi as period.

FIND THE ARCS WHOSE TRIG FUNCTIONS ARE KNOWN.

Before learning solving trig equations, you must know how to quickly find the arcs whose trig functions are known. Conversion values of arcs (or angles) are given by **trig tables** or calculators. Examples:

After solving get cos x = 0.732. Calculators give the solution arc x = 42.95 degree. The trig unit circle will give an infinity of other arcs x that have the same cos value (0.732). These values are called extended answers.

After solving get sin x = 0.5. Trig table gives the solution arc: x = Pi/6. The unit circle gives another answer: x = 5Pi/6. The extended answers are: x = Pi/6 +2K.Pi (K is a real whole number) and x = 5Pi/6 + 2k.Pi

CONCEPT IN SOLVING TRIG EQUATIONS.

To solve a trig equation, transform it into one or many basic trig equations. Solving trig equations finally results in solving 4 types of basic trig equations

KNOW HOW TO SOLVE BASIC TRIG EQUATIONS.

They are also called “trig equations in simplest forms”. There are 4 types of common basic trig equations:

sin x = a **cos x = a** (a is a given number)

tan x = a **cot x = a**

Solving basic trig equations proceeds by considering the positions of the variable arc x that rotates on the trig unit circle, and by using trig conversion tables (or calculators). To fully know how to solve these basic trig equations, or similar, see book titled:”Trigonometry: Solving trig equations and inequalities” (Amazon e-book 2010)

Example 1. Solve: sin x = 0.866

Solution. There are 2 answers given by the trig unit circle and trig conversion table:

x1 = Pi/3 x2 = 2Pi/3 Answers

x1 = Pi/3 + 2k.Pi x2 = 2Pi/3 + 2k.Pi Extended answers

Example 2. Solve: cos x = -0.5

Solution. There are 2 answers given by the unit circle, trig conversion table (or calculators):

x1 = 2Pi/3 + 2k.Pi   x2 = -2Pi/3   Answers

x2 = 2Pi/3 + 2k.Pi   x2 = -2Pi/3 + 2k.Pi   Extended answers

<u>Example 3</u>. Solve: tan (x – Pi/4) = 0

Solution. Answer given by the unit circle and calculators:

x – Pi/4 = 0 → x = Pi/4   Answer

x = Pi/4 + k.Pi   Extended answer

<u>Example 4</u>. Solve: cot 2x = 1.732

Solution. Answers given by unit circle, trig table (or calculators):

2x = Pi/6 → x = Pi/12   Answer

x = Pi/12 + k Pi/2   Extended answer

<u>Example 5.</u> Solve: sin(x – 25 deg.) = 0.5

Solution. The trig table and unit circle gives:

a. sin (x – 25 deg.) = sin 30 deg.   b. sin (x – 25 deg) = sin (180 – 30) deg.

x – 25 = 30 deg.   x – 25 deg. = 150 deg.

x = 55 deg.   x = 175 deg.

x = 55 deg. + k.360 deg.   x = 175 deg. + k.360 deg. (Extended answers)

TRANSFORMATIONS USED TO SOLVE TRIG EQUATIONS

To transform a complex trig equation into many basic trig equations, students can use common algebraic transformations (factoring, common factor, polynomials identities…), definitions and properties of trig functions, and **trig identities** (the most needed). There are about 31 trig identities, among them the last 14 identities, from # 19 to # 31, are called **transformation identities** since they are necessary tools to transform trig equations into basic ones. See above mentioned trig book (Amazon e-book 2010)

<u>Example 6</u>: Transform the sum (sin a + cos a) into a product of 2 basic trig equations:

sin a + cos a = sin a + sin (Pi/2 – a)   Use Identity "Sum into Product" (# 28)

= 2sin Pi/4.sin (a + Pi/4)   Answer

Example 7. Transform the difference (sin 2a – sin a) into a product of 2 basic trig equations, using trig identity and common factor. Use the Trig Identity: sin 2a = 2sin a.cos a.

sin 2a – sin a = 2sin a.cos a – sin a = sin a (2cos a -1)

GRAPH THE SOLUTION ARCS ON THE TRIG UNIT CIRCLE

After solving, you can graph to illustrate the solution arcs on the trig unit circle. The terminal points of these solution arcs constitute regular polygons on the trig circle. For examples:

- The terminal points of the solution arcs x = Pi/3 + k Pi/2 constitute a **square** on the unit circle.
- The solution arcs x = Pi/4 + kPi/3 are represented by the vertexes of a **regular hexagon** on the unit circle

THE COMMON PERIOD OF A TRIG EQUATION.

Unless specified, a trig equation F(x) = 0 must be solved covering one common period. This means you must find all the solution arcs within the common period of the equation. The common period of a trig equation is equal to **the least multiple** of all periods of the trig functions presented in the trig equation. Examples:

- The equation F(x) = cos x – 2tan x – 1, has 2Pi as common period
- The equation F(x) = tan x + 3cot x = 0, has Pi as common period
- The equation F(x) = cos 2x + sin x = 0, has 2Pi as common period
- The equation F(x) = sin 2x + cos x – cos x/2 = 0, has 4Pi as common period.

METHODS TO SOLVE TRIG EQUATIONS

If the given trig equation contains only one trig function of x, solve it as a basic trig equation. If the given trig equation contains two or more trig functions of x, there are two common methods for solving, depending on transformation possibilities.

1. **Method 1** – Transform the given trig equation into a **product** of many basic trig equations.

Example 8. Solve: 2cos x + sin 2x = 0   (0 < x < 2Pi)

Solution. Replace sin 2x by using the trig identity “sin 2x = 2sinx.cosx”

$$2\cos x + 2\sin x.\cos x = 2\cos x (\sin x + 1)$$

Next, solve the 2 basic trig equations: cos x = 0 and (sin x + 1) = 0

- cos x = 0 → x = Pi/2 and x = 3Pi/2
- sin x = -1 → x = 3Pi/2

Example 9. Solve the trig equation: cos x + cos 2x + cos 3x = 0 (0 < x< 2Pi)

Solution. Transform it into a product, using trig identity ”(cos a + cos b)”.

$$\cos x + \cos 2x + \cos 3x = \cos 2x (2\cos x + 1) = 0$$

Next, solve the 2 basic trig equations: cos 2x = 0 and (2cos x + 1) = 0

Example 10. Solve: sin x – sin 3x = cos 2x (0 < x < 2Pi)

Solution. Using the trig identity “(sin a – sin b)”, transform the equation into a product:

$$\sin x - \sin 3x - \cos 2x = - \cos 2x (2\sin x + 1) = 0$$

Next, solve the 2 basic trig equations: cos 2x = 0 and (2sin x + 1) = 0

Example 11. Solve: sin x + sin 2x + sin 3x = cos x + cos 2x + cos 3x

Solution. By using the “Sum into Product Identities”, and then common factor, transform this trig equation into a product:

$$\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$$
$$\sin 2x (2\cos x + 1) = \cos 2x (2\cos x + 1)$$
$$(2\cos x + 1) (\sin 2x - \cos 2x) = 0$$

Next, solve the 2 basic trig equations: (2cos x + 1) = 0 and (sin 2x – cos 2x) = 0

2. **Method 2** - If the given trig equation contains 2 or more trig functions, transform it into an equation having **only one trig function variable**. The common trig functions to choose as variable are: sin x = t; cos x = t, cos 2x = t, tan x = t; tan x/2 = t.

Example 12. Solve 3sin ^2 x – 2cos^2 x = 4sin x + 7 (1) (0 < x < 2Pi)

Solution. Replace in the equation (cos^2 x) by (1 – sin^2 x) then put the equation in standard form.

$$3\sin^2 x - 2 + 2\sin^2 x – 4\sin x – 7 = 0$$

Call sin x = t, we get: $5t^2 – 4t – 9 = 0$

This is a quadratic equation in t, with 2 real roots: t1 = -1 and t2 = 9/5. The second real root t2 is rejected since sin x must < 1. Next, solve for t = sin x = -1 → x = 3Pi/2 (Solution)

Check with the given equation (1) by replacing sin x = sin 3Pi/2

$$3 – 0 = -4 + 7$$ The solution is correct

Example 13. Solve: $\sin^2 x + \sin^4 x – \cos^2 x = 0$

Solution. Choose cos x = t as function variable.

$$(1 – t^2) (1 + 1 – t^2) – t^2 = 0$$
$$t^4 – 4 t^2 + 2 = 0$$

It is a bi-quadratic equation. There are 2 real roots: cos^2 x = t^2 = 2 + 1.414 (rejected) and

cos^2 x = t^2 = 2 – 1.414 = 0.586 (accepted since < 1).

Next solve the 2 basic trig equations: cos x = t = 0.77 and cos x = t = -0.77.

Example 14. Solve: $\cos x + 2\sin x = 1 + \tan x/2$ $(0 < x < 2\pi)$

Solution. Choose t = tan x/2 as function variable. Replace sin x and cos x in terms of tan x/2.

$$1 - t^2 + 4t = (1 + t) (1 + t^2)$$
$$t^3 + 2 t^2 – 3t = t (t^2 + 2t – 3) = 0$$

The quadratic equation (t^2 + 2t – 3 = 0) has 2 real roots: 1 and -3.

Next, solve the 3 basic trig equations: t = tan x/2 = 0 ; t = tan x/2 = -3 ; and t = tan x/2 = 1.

Example 15. Solve: $\tan x + 2 \tan^2 x = \cot x + 2$ $(-\pi/2 < x < \pi/2)$

Solution. Choose tan x = t as function variable.

$$t + 2 t^2 = 1/t + 2$$
$$(2t + 1) (t^2 – 1) = 0$$

Next, solve the 3 basic trig equations (2tan x + 1) = 0; (tan x – 1) = 0; (tan x + 1) = 0

SOLVING SPECIAL TYPES OF TRIG EQUATIONS.

There are a few special types of trig equations that require specific transformations.

Examples: a sin x + b cos x = c
a (sin x + cos x) + b cos x.sin x = c
a.sin^2 x + b.sin x.cos x + c.cos ^2 x = 0

CONCLUSION.

Solving trig equations is a tricky work that often leads to errors and mistakes. Therefore, the answers should be always carefully checked.
After solving, you may check the answers by using a graphing calculator to directly graph the given trig equation F(x) = 0.
Graphing calculators will give answers (real roots) in decimals. For example Pi, or 180 degree, is given by the value 3.14.
For more details, see the last chapter of the book titled "Solving trig equations and inequalities" (Amazon e-book 2010).

(This article was written by Nghi H. Nguyen, co-author of the new Diagonal Sum Method for solving quadratic equations)