Solving Quadratic Equations by the new improved Factoring "AC Method"

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So far, the "ac method" (YouTube.com) has been the best factoring method to solve a quadratic equation in general form ax^2 + bx + c = 0. Another factoring method, called the "Box Method" (YouTube. com) offers a similar solving approach.

THE FACTORING "AC METHOD".

Given a factorable quadratic equation in standard form ax^2 + bx + c = 0. This method aims to factor the quadratic equation into 2 binomials in x by replacing in the equation the term (bx) by the 2 terms (b1x) and (b2x) that satisfy these 2 conditions:

1. The product b1.b2 = ac
2. The sum (b1 + b2) = b.

Example 1. Solve: 5x^2 + 6x – 8 = 0.

Solution. Find 2 numbers that the product is ac = -40 and the sum is (b = 6). We proceed by composing all factors of ac = -40: [(-1, 40)(1, -40)(-2, 20)(2, -20)(-4, 10).OK].

Next, substitute the term (6x) in the equation by the 2 terms (-4x) and (10x).

5x^2 – 4x + 10x – 8 = 5x(x + 2) – 4(x + 2) = (x + 2) (5x – 4) = 0.

Then, solve the 2 binomials for x:

(x + 2) = 0 → x = -2

(5 – 4) = 0 → x = 4/5

Example 2. Solve 15x^2 – 53x + 16 = 0.

Solution. Find 2 numbers that the product is ac = 15 x 16 = 240 and the sum = -53. Proceeding: [(1, 240),(-1, -240),(2, 120),(-2, -120),(3, 80),(-3, -80),(4, 60),(-4, -60),(5, 48),(-5, -48). OK]. Next, replace in the equation the term (-53x) by the 2 terms (-5x) and (-48x), then put terms into common factors.

15x^2 -5x – 48x + 16 = 0 = 5x(3x – 1) - 16(3x – 1) = (3x -1)(5x – 16) = 0

Next, solve the 2 binomials:

3x – 1 = 0 → x = 1/3
5x – 16 = 0 → x = 16/5

Remark. In the solving approach, in order to get the product **ac** = 240, we have to compose both factor pairs (1, 240) and (-1, -240). Then, we do the same thing for (2, 120) and (-2, -120) and so on. Is it possible to simplify the process if we know in advance the signs of the 2 real roots? Yes, if we apply **The Rule of Signs For Real Roots** of a quadratic equation into the solving approach.

RECALL THE RULE OF SIGN FOR REAL ROOTS.

a. When **a** and **c** have opposite signs, both real roots have opposite signs.

Example. The equation $5x^2 + 8x - 13 = 0$ has 2 real roots with opposite signs: 1 and -13/5
Example. The equation $7x^2 – 8x – 15 = 0$ has 2 real roots with opposite signs: (-1) and (15/7)

b. When **a** and **c** have the same sign, both real roots have the same sign.

- If **a** and **b** have opposite signs, both real roots are positive.

Example: The equation $8x^2 - 11x + 3 = 0$ has 2 real roots both positive: 1 and 3/8.
Example: The equation $13x^2 – 21x + 8 = 0$ has 2 real roots both positive: 1 and 8/13

- If **a** and **b** have the same sign, both real roots are negative.

Example: The equation $8x^2 + 11x + 3 = 0$ has 2 negative real roots: (-1) and (-3/8).
Example: The equation $13x^2 + 23x + 10 = 0$ has 2 negative real roots: (-1) and (-10/13).

THE NEW IMPROVED FACTORING "AC METHOD".

Its approach is to factor the given quadratic equation into 2 binomials in x by replacing in the equation the term (bx) by the 2 terms (b1x) and (b2x) that satisfy these conditions:

1. b1.b2 = ac
2. b1 + b2 = b
3. Application of the Rule of Signs to find b1 and b2.

A. When a = 1. Solving quadratic equation type $x^2 + bx + c = 0$

When a = 1, the product **ac = c**. Solving becomes solving the popular puzzle: "Find 2 numbers knowing their product (c) and their sum (-b)". We can directly get the real roots of the quadratic equation by applying the Rule of Sign when composing the factors of c. The factor pair, whose sum matches (b), or (-b), gives the answers. There is no need for factoring.

a. When **a** and **c** have opposite signs, roots have opposite signs. First, proceed composing factors of **ac = c** and in the same time apply the Rule of Sign. By convention, always put the negative sign (-) in front of the first number inside the factor pair. We will see later that this doesn't affect the outcome at all
.
b. When **a** and **c** have same sign, both real roots have same sign. By convention, proceed composing factor pairs of **ac = c** with positive (+) numbers only. This will not affect the outcome at all.

Example 3. Solve: $x^2 - 11x - 102 = 0$.

Solution. Since a and c have opposite signs, roots have opposite signs. Compose the factors of ac = c = -102. By convention, when roots have opposite signs, we can put the negative sign (-) in front of the first number of the factor pair.

Proceeding: (-1, 102)(-2, 51)(-3, 34)(-6, 17). OK!.
The sum of the last pair is: 11 = -b. Consequently, these numbers (-6 and 17) are the 2 real roots of the given quadratic equation. There is no need for factoring and solving the 2 binomials for x.

Note. If we proceed composing factors of ac = c differently, the outcome would be the same.
Proceeding: (1, -102)(2, -51)(3, -34)(6, -17). OK!
The sum of the last pair is (-11) = b. Consequently, the 2 real roots make the negative of this pair. They are: -6 and 17.

Example 4. Solve $x^2 + 31x + 108 = 0$

Solution. Both roots are negative. Proceed to find b1 and b2 with + numbers only inside the factor pairs. Proceeding: (1, 108)(2, 54)(3, 36)(4, 27). OK! This sum (4 + 27) = 31 = b. Since the sum of the 2 real roots is –b, then the 2 real roots are: -4 and -27. There is no need for factoring.

Note. If we proceed differently by composing factors of c = 108 with all negative numbers inside the factor pairs, the outcome will be the same.
Proceeding: (-1, -108)(-2, -54)(-3, -36)(-4, -27). OK!
The sum of the last pair is (-4 – 27) = - 31 = -b. Then, the 2 real roots are: -4 and -27. No factoring!

Example 5. Solve: $x^2 - 27x + 126 = 0$.

Solution. Both roots are positive. Compose the factors of ac = c = 126 with all positive numbers.
Proceeding: (1, 126)(2, 63)(3, 42)(6, 21). OK!
The sum of the last pair is: (6 + 21) = 27 = -b. Then, the 2 real roots are: 6 and 21. No factoring needed.

B. When a is not 1 - Solving quadratic equation type $ax^2 + bx + c = 0$

In these cases, the factoring AC Method finds the 2 numbers b1 and b2 that satisfy the 2 conditions: the product (b1.b2) = ac, and the sum (b1 + b2) = b. The application of the Rule of Signs leads to these following practices:

1. If the 2 roots have opposite signs, compose the factor pairs of the product **ac** with all the first numbers of the pair being negative.

2. If the 2 roots have same sign, compose the factor pairs of the product **ac** with all positive numbers.

Example 6. Solve: $8x^2 - 22x - 13 = 0$.

Solution. Roots have opposite signs. Find 2 numbers that the product is ac = -104, and the sum b1 + b2 = b = -22. Compose factors of **ac** with first number of the pair being negative.
Proceeding: (-1, 104),(-2, 52),(-4, 26). OK!

This last sum is 22 = -b. Then: b1 = 4 and b2 = -26.
Next, replace into the equation the term (-22x) by the 2 terms (4x) and (-26x).

$8x^2 – 4x + 26x – 13 = 4x(2x + 1) – 13(2x + 1) = (2x +1) (4x – 13) = 0$
Then, solve the 2 binomials for x:
$2x + 1 = 0 \rightarrow x = -1/2$
$4x – 13 = 0 \rightarrow x = 13/4$

<u>Example 7</u>. Solve: $12x^2 + 5x – 72 = 0$.

Solution. Roots have opposite signs. Find 2 numbers that the product is ac = -864.
Proceeding: (-1, 864)(-2, 432)(-3, 288)(-4, 216)(-6, 144)(-8, 108)(-12, 72) (-16, 54)(-18, 48)(-24, 36)(-32, 27). OK. This sum is (-32 + 27) = -5 = -b. Then (b1 + b2) = b = (32 – 27).
Next, replace the term (5x) by the 2 terms (32x) and (-27x).

$12x^2 + 32x – 27x – 72 = 0$
$4x (3x + 8) – 9(3x + 8) = 0$
$(3x + 8)(4x – 9) = 0$.

$3x + 8 = 0 \rightarrow x = -8/3$
$4x – 9 = 0 \rightarrow x = 9/4$

<u>Example 8.</u> Solve $20x^2 – 59x – 16 = 0$.

Solution. Find b1 and b2 knowing their product is ac = 320 and their sum is -59. Roots have opposite signs.
Proceeding: (-1, 320)(-2, 160)(-4, 80)(-5, 64). OK. This last sum is 59 = -b.
Consequently, b1 = 5 and b2 = -64. Replace in the equation the term (-59x) by the 2 terms (5x) and (-64x).
$20x^2 + 5x – 64x – 16 = 5x(4x + 1) - 16(4x + 1) = (4x + 1)(5x – 16) = 0$

$4x + 1 = 0 \rightarrow x = -1/4$
$5x – 16 = 0 \rightarrow x = 16/5$.

<u>Example 9.</u> Solve: $24x^2 + 59x + 36 = 0$.

Solution. Both roots are negative. Product b1.b2 = 864. Sum (b1 + b2) = 59.
Proceeding with **positive numbers** only: (1, 864)(2, 432)(4, 216)(6, 1440(8, 108)(9, 96)(12, 72)(16, 54)(18, 48)(24, 36)(27, 32). OK. This sum (27 + 32) = 29 = b.
Next, replace in the equation the term (59x) by the 2 terms (27x) and (32x).

$$24x^2 + 27x + 32x + 36 = 8x(3x + 4) + 9(3x + 4) = (3x + 4)(8x + 9) = 0$$
$3x + 4 = 0 \rightarrow x = -4/3$
$8x + 9 = 0 \rightarrow x = -9/8$

Example 10. Solve $12x^2 + 29x + 15 = 0$.

Solution. Both real roots are negative. Compose factors of ac = 180 with all positive numbers. Proceeding: (1, 180)(2, 90)(3, 60)(4, 45)(5, 36)(6, 30)(9, 20). OK!
This last sum is 29 = b. Then b1 = 9 and b2 = 20. Next, replace in the equation the term (29x) by the 2 terms (9x) and (20x).
$12x^2 + 9x + 20x + 15 = 3x(4x + 3) + 5(4x + 3) = (4x + 3)(3x + 5) = 0$

$4x + 3 = 0 \rightarrow x = -3/4$
$3x + 5 = 0 \rightarrow x = -5/3$

Example 11. Solve: $16x^2 – 55x + 21 = 0$.

Solution. Both roots are positive. Find 2 numbers that product is ac = 336, and the sum (b1 + b2) = b = -41. Compose factors of ac with all numbers positive.
Proceeding: (1, 336)(2, 168)(4, 82)(6, 56) (7, 48). OK. This sum is 7 + 48 = 55 = -b. Then, b1 = -7 and b2 = -48.
Next, replace in the equation the term (-55x) by the 2 terms (-7x) and (-48x).

$$16x^2 – 48x – 7x + 21 = 16x(x – 3) - 7(x - 3) = (x - 3)(16x - 7) = 0$$

$x – 3 = 0 \rightarrow x = 3$
$16x - 7 = 0 \rightarrow x = 7/16$.

Example 12. Solve $12x^2 – 53x + 20 = 0$

Solution. Both real roots are positive. Compose factors of ac = 240 with all positive numbers.
Proceeding: (1, 240(2, 120)(3, 80)(4, 60)(5, 48). OK!
The sum of the last pair is 53 = -b. Then, b1 = -5 and b2 = -48.
Replace in the equation (-53x) by the 2 terms (-5x) and (-48x).
$12x^2 – 48x – 5x + 20 = 12(x-4) – 5(x -4) = (x – 4)(12x – 5)= 0$

$x – 4 = 0 \rightarrow x = 4$
$12x – 5 = 0 \rightarrow x = 5/12$

IMPORTANT TIPS TO REMEMBER

- **1.** When the 2 real roots have **opposite signs** (**a** and **c** opposite signs), by convention, always compose factor pairs of the product **ac** with all the first number of the pairs being negative. Next, carefully take into consideration the sign of the sum (b1 + b2) comparing to **(b)**, or (**-b**). Then, proceed to replace in the equation the term **(bx)** by the 2 terms **(**b1x**)** and **(**b2x**)** as usual.

- **2.** When the roots have **same sign** (**a** and **c** same sign), list the factor pairs of **ac** with all **positive** numbers inside. Next, carefully take into consideration the sign of the sum (b1 + b2) comparing to **(b)** or **(-b).**

CONCLUSION AND COMMENTS.

We can improve the factoring "AC Method" by applying the Rule of Signs For Real Roots into the solving approach. This **new and improved Factoring "AC Method"** helps:

1. To know in advance the 2 signs **(+ or -)** of the 2 real roots for a better selection of the solving approach.

2. Simplify the solving process by reducing the number of permutations in half when the roots have opposite signs. For example, when composing the factor pairs of ac = -24, instead of listing all the **8 pairs**: (-1, 24)(1, -24)(-2, 12)(2, -12)(-3, 8)(3, -8)(-4, 6)(4, -6), we only need to list half of them, or **4 pairs**: (-1, 24)(-2, 12)(-3, 8)(-4, 6).

3. Simplify the solving process, when roots have same sign, by composing factors of the product ac with positive numbers only. For example, when composing the factors of c = 24, we only list factor pairs with positive numbers inside: (1, 24)(2, 12)(3, 8)(4, 6) and ignore the pairs: (-1, -24)(-2, -12)(-3, -8)(-4, -6).

4. In case when **a = 1**, the 2 real roots can be immediately obtained from the list of the factor pairs of **c.** Factoring the quadratic equation and solving the 2 binomials are useless in this case.

(This article titled "Solving Quadratic Equations by the new improved Factoring AC Method" was written by Nghi H. Nguyen, co-author of the Diagonal Sum Method for solving quadratic equation)