BEST METHODS FOR SOLVING QUADRATIC INEQUALITIES.

I. GENERALITIES

There are 3 common methods to solve quadratic inequalities. Therefore, students sometimes are confused to select the fastest and the best solving method. I generally explain below these 3 methods and then compare them through selected examples.

Solving a quadratic inequality, in standard form f(x) = ax^2 + bx + c > 0 (or < 0), means finding all the values of x that make the inequality true.
These values of x constitute the **solution set** of the inequality.
The solution set of a quadratic inequality are expressed in the form of **intervals.**
Examples of quadratic inequalities:

x^2 − 8 x + 7 < 9	3x^2 + 4x − 7 < 0	5x^2 − 12x > 17
(3x − 5)(4x + 1) < 0	3x/(x -1) + 4x/(3 − x) > 1	1/(x − 2) + 2/(x − 3) > 2

Examples of solution set expressed in terms of intervals:
(-2, 3)   (1.73, 3.42)   (-infinity, 3]   [-1/3, 13/5]   [4, +infinity)

II. STEPS IN SOLVING QUADRATIC INEQUALITIES

In general, there are 4 common steps.

Step 1. Transform the inequality into standard form: f(x) = ax^2 + bx + c < 0 (or > 0).

Example. Solve: x(4x -3) > 7.

In first step, transform the inequality into standard form: f(x) = 4x^2 − 3x − 7 > 0.

Example. Solve: (2x + 5)(x − 1) < -3

In first step, transform the inequality into standard form: f(x) = 2x^2 + 3x − 2 < 0.

Step 2. Solve the quadratic equation f(x) = 0. You may use one of the 4 existing common methods (factoring ac method, completing the square, quadratic formula, graphing) or the new Diagonal Sum Method (Amazon e-book 2010).

Before starting, find out if the equation has 2 real roots. How? Roughly estimate the Discriminant D's value. You don't need to calculate its exact value, unless you want to apply the quadratic formula. Just use mental math to see if D is positive (> 0) or negative (< 0).

If **D < 0**, or **D = 0,** solve the inequality by referring to the first part of the Theorem explained in next paragraph.

If **D > 0**, solve the equation f(x) = 0 to get the 2 real roots x1 and x2. Find out if this equation can be factored. How? You may calculate D to see if it is a perfect square. Or, you may first use the new Diagonal Sum Method to solve the equation. It usually requires fewer than 3 trials. If it fails, meaning no diagonal sums is equal to b (or –b), then the equation can't be factored, and therefore the quadratic formula must be used.

Step 3. Solve the given quadratic inequality f(x) < 0 (or > 0), based on the 2 values x1 and x2, found in Step 2. You may choose one of the 3 common methods to solve quadratic inequalities described below.

Step 4. Express the solution set of the quadratic inequality in terms of **intervals**. You must know how to correctly use the interval symbols. Examples:

(2, 7): **open** interval between 2 and 7. The 2 end (critical) points are not included in the solution set.

[-3, 5]: **closed** interval. The 2 end points -3 and 5 are included in the solution set.

[2, +infinity): **half-closed** interval; only the end point 2 is included in the solution set

(-infinity, -1]: **half closed** interval; only the end point is included in the solution set.

III. COMMON METHODS TO SOLVE QUADRATIC INEQUALITIES

There are 3 most common methods. Therefore, students are sometimes confused to select the best solving method.

1. **The number line and test point method.**

   Given a quadratic inequality in standard form f(x) = ax^2 + bx + c < 0 (or > 0), with **a** not equal zero.
   Suppose the Discriminant **D** > 0, and the given quadratic equation has 2 real roots x1 and x2. Plot them on a number line. They divide the number line into one segment (x1, x2) and 2 rays. The solution set of the quadratic equation should be either the segment, or the 2 rays. Always use the **origin O** as test point. Substitute x = 0 into the inequality. If it is true, then the origin O is located on the true segment (or the true ray). If one ray is a part of the solution set, then the other ray also belongs to the solution set, due to the symmetrical property of the parabola graph.

   Note 1. When D < 0, there are no real roots, this number line method can't be used. In this case you must solve the inequality by the algebraic method.
   When D = 0, there is a double root at x = -b/2a, this number line method can't be used. You must apply the algebraic method.
   Note 2. By this number-line method, you may use a double number line, or even a triple number line, to solve a system of two or three quadratic inequalities in one variable. See book titled "New methods for solving quadratic equations and inequalities" (Amazon e-book 2010).

Examples of solving by the test point method.

Example. Solve: $x^2 - 15x < 16$.
Solution. First step, write the inequality in standard form $f(x) = x^2 - 15x - 16 < 0$.
Second step, solve $f(x) = 0$. The 2 real roots are -1 and 16.
Third step, solve the inequality $f(x) < 0$. Plot the 2 real roots -1 and 16 on a number-line. The origin O is located inside the segment (-1, 16). Use the origin O as test point. Substitute $x = 0$ into the inequality. We get -16 < 0. It is true, then the origin O is located on the true segment.
Step 4, express the solution set in the form of open interval (-1, 16). The 2 endpoints -1 and 16 are not included in the solution set.

Example. Solve: $-3x^2 < -8x + 5$
Solution. First step, write the inequality into the standard form: $f(x) = -3x^2 + 8x - 5 < 0$.
Second step, solve $f(x) = 0$. The 2 real roots are 1 and 5/3.
Third step, plot these values on a number line. The origin O is located on the left ray. Use O as test point. Substitute $x = 0$ into the inequality. We get: -5 < 0. It is true, then the origin O is located on the true ray. By symmetry, the other ray also belongs to the solution set. Last step, express the solution set in the form of intervals:
(-infinity, 1) and (5/3, +infinity). The 2 end points 1 and 5/3 are not included.

Example. Solve: $9x^2 < 12x - 1$
Solution. First step: $f(x) = 9x^2 - 12x + 1 < 0$.
Second step: Solve $f(x) = 0$. The new Diagonal Sum Method fails to solve it, this equation can't be factored. We must use the quadratic formula. The 2 real roots are x1 = (2 − 1.73)/3 = 0.09 and x2 = (2 + 1.73)/3 = 1.24.
Third step: Plot the numbers on a number line. The origin is located on the left ray. Substitute $x = 0$ into the inequality. We have: 1 < 0. It is not true. The origin O is not on the true ray. The solution set is the segment (0.09, 1.24).
Step 4, solution set: open interval (0.09, 1.24); the end points not included.

2. **The algebraic method.**

This solving method is popular in Europe. It is based on a theorem about the sign status of a trinomial $f(x) = ax^2 + bx + c$, with a not zero, and $D = b^2 - 4ac$

Theorem on the sign status of a trinomial f(x).

a. If **D < 0**, f(x) has the same sign as **a** regardless of the values of x.

   Example 1. The trinomial $f(x) = 3x^2 - x + 7$ has $D = b^2 - 4ac = 1 - 84 = -83 < 0$.
   This trinomial f(x) is always positive, same sign as a = 3, regardless of the values of x.

<u>Example 2.</u> The trinomial $f(x) = -5x^2 + 3x - 8$ has $D = 9 - 160 = -151 < 0$. This $f(x)$ is always negative (< 0), same sign as $a = -5$, regardless of the values of x.

b. When **D = 0**, f(x) has the same sign as **a** for any values of x different to $(-b/2a)$.

c. When **D > 0**, f(x) has the opposite sign of **a** between the 2 real roots x1 and x2, and f(x) has the same sign as **a** outside the interval (x1-x2).

   <u>Example</u> . The trinomial $f(x) = x^2 - 8x - 9$ has $D = 64 + 36 = 100 = 10^2 > 0$. The equation $f(x) = 0$ has 2 real roots (-1) and (9). The trinomial f(x) is negative (< 0) within the interval (-1, 9). f(x) is positive (> 0) outside this interval.

   <u>Example</u> . The trinomial $f(x) = -x^2 + 5x - 4 = 0$ has $D = 25 - 16 = 9 = 3^2 > 0$. The equation $f(x) = 0$ has 2 real roots 1 and 4. The trinomial f(x) is positive, opposite to the sign of a = -1, within the interval (1, 4).

The Theorem's proof

a. **When D < 0**. We can write f(x) in the form $f(x) = a(x^2 + bx/a + c/a)$. **(1)**
   Recall the equation developed to find the quadratic formula:

   $(x^2 + bx/a + cx/a) = 0$
   $x^2 + bx/a + b^2/4a^2 - b^2/4a2 + c/a = 0$
   $(x + b/2a)^2 - (b^2 - 4ac)/4a^2 = 0$ (Call $D = b^2 - 4ac$).
   $(x + b/2a)^2 - D/4a^2 = 0$

Substitute this relation into the equation **(1).**

   $f(x) = a[(x + b.2a)^2 - D/4a^2]$ **(2)**

 a.   When D < 0, the quantity inside the parenthesis is always positive. Therefore, f(x) has the same sign as **a** regardless of the values of x.

b. **When D = 0**, the equation (2) reduces to: $f(x) = a(x + b/2a)^2$.
   We see that f(x) has the same sign as **a** regardless of the values of x, since the quantity $(x + b.2a)^2$ is always positive.

c. **When D > 0**. There are 2 real roots x1 and x2, with their sum $x1 + x2 = -b/a$, and their product $x1.x2 = c/a$.
   We can write f(x) in the form $f(x) = a(x^2 + bx/a + c/a)$. The quantity in parenthesis is a quadratic equation that can be factored into 2 binomials in x with x1 and x2 as real roots.   Page 4 of 7

$f(x) = a(x - x1)(x - x2)$ **(1)**

Now, setup a **Sign Table** (sign chart) to study the sign status of f(x) when x varies from −infinity to +infinity and passes by the 2 values x1 and x2. Suppose x2 > x1.

x	-infinity		x1		x2		+infinity
x − x1		-	0	+		+	
x − x2		-		-	0	+	
(x − x1)(x − x2)		+	0	-	0	+	

We see from the relation **(1)** and the last line of the chart that f(x) has the opposite sign of **a** between the 2 real roots x1 and x2. Outside the interval (x1, x2), f(x) have the same sign as **a**.

Graphic interpretation of the Theorem.

We can easily understand this Theorem by looking at the parabola graph of a quadratic equation.
If a > 0, the parabola is **upward**. Between the two x-intercepts (real roots) of the parabola, a part of the parabola is **below** the x-axis, meaning f(x) is negative (< 0) within this interval (x1, x2), opposite in sign to a (positive).
If a < 0, the parabola is **downward**. Between the two x-intercepts (real roots), a part of the parabola is **above** the x-axis, meaning f(x) is positive within this interval (x1, x2), opposite in sign to a (negative).

Examples about the Theorem.

Example . The graph of the trinomial f(x) = x^2 − 7x − 8 is an upward parabola whose x-intercepts are at -1 and 8. Within the interval (-1, 8) a part of the parabola is below the x-axis, meaning f(x) is negative (< 0) within this interval, opposite in sign to **a** (positive)

Example . The graph of f(x) = - x^2 + 11x − 10 is a downward parabola whose x-intercepts are at 1 and 10. Within the interval (1, 10), a part of the parabola is above the x-axis, meaning f(x) is positive (> 0) in this interval, opposite in sign to **a** (< 0).

NOTE. Students study once the theorem's development, then they apply it to solve various types of quadratic inequalities.

3. **The graphing method**

The concept of this method is simple. When the parabola graph of the trinomial f(x) is above the x-axis, the trinomial f(x) is positive, and vice-versa.
You don't need to accurately graph the parabola. Based on the 2 real roots obtained by solving f(x) = 0, you may just draw a rough sketch of the parabola.
Pay only attention to if the parabola is upward or downward, by considering the sign of **a**. After graphing, you get the solution set by looking at the graph of the parabola. Last, express the solution set in the form of intervals.
By this graphing method, you may solve a system of two, or even three, quadratic inequalities, by graphing them on the same coordinate grid.

IV. EXAMPLES OF SOLVING QUADRATIC INEQUALITIES

<u>Example</u>. Solve: $f(x) = 13x^2 - 7x + 8 > 0$.
Solution. The Discriminant D = 49 − 416 < 0. There are no real roots. Use the algebraic method (Theorem) to solve it. Since a is positive, f(x) is always positive (> 0) regardless of the values of x. The inequality is always true.

<u>Example</u>. Solve: $f(x) = -5x^2 + 6x - 10 > 0$.
Solution. D = 36 − 200 < 0. There are no real roots. Use the Theorem to solve. The trinomial f(x) is always negative (< 0), same sign as a = -5, regardless of x. The equation is always not true.

<u>Example</u>. Solve: $f(x) = 3x^2 - 7x - 10 > 0$.
<u>Solution by the number line and test point method</u>. The equation f(x) = 0 has 2 real roots -10/3 and 1. Plot these numbers on a number line. The origin O is located inside the segment (-10/3, 1). Substitute x = 0 into the inequality. We have: 10 > 0. It is not true, then O is not on the solution set. The solution set is the 2 rays. Express the solution set by intervals: (-infinity, -10/3) and (1, +infinity).
<u>Solution by the algebraic method</u>. The equation f(x) has 2 real roots -10/3 and 1. Inside the segment (-10/3, 1) f(x) is negative (< 0), opposite in sign to a = 5. Therefore the 2 rays are the solution set. Answers: (-infinity, -10/3) and (1, +infinity). There is no need to draw the number line and doing the test point.

<u>Example</u>. Solve: $f(x) = -3x^2 + 18x - 15 < 0$.
<u>Solution by the number line</u>. The equation f(x) = 0 has 2 real roots 1 and 5. Plot these numbers on a number line. The origin O is outside the segment (1, 5). Substitute x = 0 into the inequality. We get: -15 < 0. It is true then O is located on the true ray. The 2 rays are the solution set. Answers: (-infinity, 1) and (5, +infinity).
<u>Solution by the algebraic method.</u> The 2 real roots are 1 and 5. Outside the interval (1, 5) f(x) is positive (> 0), opposite in sign to a = -3. The 2 rays are the solution set.
Answers: (-infinity, 1) and (5, +infinity).

V. REMARKS AND CONCLUSION

Remark 1. When the inequality f(x) < 0 (or > 0) has an additional equal (=) sign, the 2 end points are automatically included in the solution set.
Remark 2. When **D < 0** and **D = 0**, the number line and test point method can't be used. The algebraic method (Theorem) must be applied.
Remark 3. When **D > 0**, there are 2 real rots. Plotted these roots on the number line, they divide it into 3 areas. Taking the origin O as test point in one area is enough. It is useless to have 3 test points for 3 areas. Using 3 test points for 3 areas is effective for first-time lesson explaining, but it is useless when students have mastered the solving process.
During tests/exams, time is limited. Students must find answers as fast as possible. Taking 3 test points would cost too much precious time.
Remark 4. There is another algebraic method that discusses the sign of the 2 binomials (x − x1) and (x − x2). This method is unadvised because it takes too much time and it easily leads to error/mistakes.

Example. Solve: f(x) = 9x^2 − 8x - 17 < 0.
Solution. The equation f(x) = 0 has 2 real roots (-1) and (17/9). Create a Sign Table with x varies from –infinity to +infinity and passes by the 2 real roots (-1) and (17/9).

x	-infinity		-1	17/9	+infinity
x + 1		-	0	+	+
x − 17/9		-	-	0	+
f(x)		+	0	-	0	+

The solution set of f(x) < 0 is the open interval (-1, 17/9).

Conclusion.

The **algebraic method** may be the **best** one to solve quadratic inequalities.
It is fast because it doesn't require to draw the number line and to take the test point each time. Just by considering the **sign** of **a**, we immediately know in which intervals **f(x)** would be **positive** (or negative).
In addition, it helps solving quadratic inequalities when **D < 0** or when **D = 0**.
For multiple choice answers in tests/exams, this method may be the **best** one.

(This article was written by Nghi H. Nguyen, co-author of the new Diagonal Sum Method for solving quadratic equations)