A METHOD FOR SOLVING SYMMETRIC INEQUALITIES.

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The method is applicable to certain broad categories of symmetric inecuații. We have chosen as an example a few inecuații appeared in collections and in [1].

1. If a,b.c are real numbers such that $a + b + c = 1$, then $15(ab+bc+ca)-27abc \le 4$.

Solution. The Inequality $15(ab+bc+ca)-27abc \le 4$ it is equivalent to $\frac{5}{9}(ab+bc+ca) - abc \le \frac{4}{27}$.

Let f be a polynomial which has roots a, b, c, we have $f(t)=(t-a)(t-b)(t-c) = t^3-(a+b+c)t^2 +(ab+bc+ca)t-abc$. If $a + b + c = 1$ then we have $f(t)=(t-a)(t-b)(t-c) = t^3-t^2 +(ab+bc+ca)t - abc$. Taking into account the inequality C-B-S one obtains $t^3-t^2 +(ab+bc+ca)t-abc \le \frac{1}{27}[3t - (a+b+c)]^3$. For $t =\frac{5}{9}$ one obtains $(\frac{5}{9})^3 - (\frac{5}{9})^2 +(ab+bc+ca)\frac{5}{9}-abc \le \frac{1}{27}(\frac{2}{3})^3$ where it appears $\frac{5}{9}(ab+bc+ca) - abc \le \frac{4}{27}$. Egalitatea se obține pentru $a = b = c =\frac{1}{3}$.

2. If a, b, c are real numbers such that $ab + bc + ca = 2$, then

$$2abc(a+b+c) - (abc)^2 \le \frac{64}{27}.$$

Solution. Let f be a polynomial that has roots, bc, ab ca, then $f(t)=(t-ab)(t-bc)(t-ca) = t^3-(ab+bc+ca)t^2 +abc(a+b+c)t-(abc)^2$. If $ab + bc + ca = 2$ then we have $f(t)=(t-ab)(t-bc)(t-ca) = t^3-2t^2 + abc(a+b+c)t-(abc)^2$. Taking into account the inequality C-B-S one obtains $t^3-2t^2 + abc(a+b+c)t-(abc)^2 \le \frac{1}{27}[3t-(ab+bc+ca)]^3$. For t =2 one obtains

$8 - 8 + abc(a+b+c)t-(abc)^2 \le \frac{1}{27}(4)^2$ where it appears $2abc(a+b+c) - (abc)^2 \le \frac{64}{27}$.

Equality is obtained for $a = b = c =\frac{\sqrt{6}}{3}$.

3. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 4$, then

$$\frac{5}{3}(a^2b^2 + b^2c^2 + c^2a^2) - (abc)^2 \le \frac{176}{27}.$$

Solution. Let f be a polynomial that has roots a^2 ,b^2 ,c^2, then $f(t)=(t-a^2)(t-b^2)(t-c^2) = t^3-(a^2 + b^2 + c^2)t^2 +(a^2b^2 + b^2c^2 + c^2a^2)t-(abc)^2$. If $a^2 + b^2 + c^2 = 2$ then we have $f(t)=(t-a^2)(t-b^2)(t-c^2) = t^3-4t^2 +(a^2b^2 + b^2c^2 + c^2a^2)t-(abc)^2$. Taking into account the inequality C-B-S one obtains $t^3-4t^2 + abc(a+b+c)t-(abc)^2 \le \frac{1}{27}[3t-(a^2 + b^2 + c^2)]^3$. For $t =\frac{5}{3}$ one obtains

$\frac{125}{27} - \frac{100}{9} + \frac{5}{3}$ abc(a+b+c)t− $(abc)^2 \le \frac{1}{27}$ where it appears $\frac{5}{3}$ abc(a+b+c)t− $(abc)^2 \le \frac{176}{27}$.

Equality is obtained for a = b = c = $\frac{\sqrt{12}}{3}$.

4. Either a, b, c three strictly positive real numbers that satisfy the condition:

$$ab + bc + ca + 2abc = 1. \qquad (1)$$

To point out that:

a) ab +bc+ca $\ge \frac{3}{4}$; b) a + b + c $\ge \frac{3}{2}$; c) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 4(a + b + c)$

Solution.

a) From $1 = ab + bc + ca + 2abc \ge 4\sqrt[4]{2a^3b^3c^3} \Leftrightarrow a^3b^3c^3 \le \frac{1}{512} \Leftrightarrow abc \le \frac{1}{8}$. (2)

Observation. At the dam for the junior Balkan Olympiad was given the following inequality: Either x, y, z three strictly positive real numbers such that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 2$. To show that $8xyz \le 1$. How the relation is equivalent to xy +yz + zx + 2xyz =1 the relationship (2) is a solution of the problem.

From (1) we have $abc = \frac{1-ab-bc-ca}{2}$ and substituting in the (2) one obtains

$\frac{1-ab-bc-ca}{2} \le \frac{1}{8}$ where it appears ab +bc+ca $\ge \frac{3}{4}$. (3)

b) Let f be a polynomial which has roots a,b,c, then $f(t)=(t-a)(t-b)(t-c) = t^3-(a+b+c)t^2 +(ab+bc+ca)t-abc \Leftrightarrow 2t^3-2(a+b+c)t^2 +2(ab+bc+ca)t-2abc = 2(t-a)(t-b)(t-c)$.

For $t = -\frac{1}{2}$ one obtains $-\frac{1}{4} - \frac{1}{2}(a+b+c) - (ab+bc+ca + 2abc) = 2(-\frac{1}{2}-a)(-\frac{1}{2}-b)(-\frac{1}{2}-c)$

$\Leftrightarrow -\frac{1}{4} - \frac{1}{2}(a+b+c) - 1 = 2(-\frac{1}{2}-a)(-\frac{1}{2}-b)(-\frac{1}{2}-c) \Leftrightarrow \frac{5}{4}+\frac{1}{2}(a+b+c) = 2(\frac{1}{2}+a)(\frac{1}{2}+b)(\frac{1}{2}+c)$

$\Rightarrow \frac{5}{4}+\frac{1}{2}(a+b+c) \le \frac{2}{27}(\frac{3}{2}+a+b+c)^3 \Leftrightarrow \frac{5}{4}+\frac{1}{2}(a+b+c) \le \frac{2}{27}\left[\frac{27}{8}+\frac{27}{4}(a+b+c)+\frac{9}{2}(a+b+c)^2+(a+b+c)^3\right] \Leftrightarrow 2(a+b+c)^3 + 9(a+b+c)^2 - 27 \ge 0 \Leftrightarrow (a+b+c+3)^2[2(a+b+c)-3]$

≥ 0 where it appears a + b + c $\ge \frac{3}{4}$. Equality is obtained for a = b = c = $\frac{1}{2}$.

c) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 4(a+b+c) \Leftrightarrow ab+bc+ca \ge 4abc(a+b+c)$ (4)

Let f be a polynomial which has roots ab,bc,ca, then $f(t)=(t-ab)(t-bc)(t-ca) = t^3-(ab+bc+ca)t^2 + (a+b+c)abct - a^2b^2c^2$. We do $t = \frac{1}{4}$ and we note with A = ab +bc+ca $- 4abc(a+b+c)$, B = ab+bc+ca and with C = abc and one obtains

$\frac{1}{64} - \frac{1}{16}$ A − a^{2}b^{2}c^2 ≤ $\frac{1}{27}$ $(\frac{3}{4} - B)^3$ ⟺ A ≥ $\frac{1}{4}$ −16 a^{2}b^{2}c^2 − $\frac{16}{27}$ $(\frac{3}{4} - \text{B})^3$ ⟺ A ≥ $\frac{16B^3-144B^2+243B-108}{27}$ ⟺ A ≥ $\frac{1}{27}$(4B^2−33B+36)(4B−3) . Because B = 1 – 2C one obtains A ≥ $\frac{4}{27}$(4C^2+5C+1)(4B−3).

Results ab +bc+ca − 4abc(a + b + c) ≥ $\frac{4}{27}$[4(abc)^2+5abc+1][4(ab+bc+ca−3].

Taking into account the (3) one obtains ab +bc+ca − 4abc(a + b + c) ≥ 0.

Equality is obtained for a = b = c = $\frac{1}{2}$.

The results obtained are solving a problem at O.B.M. J 2006 and a problem proposed by Mircea Lascu and Marian Tetiva from [1].

5. Either x, y, z with positive real numbers such that:

$$x^2 + y^2 + z^2 + 2xyz = 1. \quad (1)$$

To show that: a) x + y + z ≤ $\frac{3}{2}$; b) xy + yz + zx ≤ $\frac{3}{4}$ ≤ x^2 + y^2 + z^2; c) x^2 + y^2 + z^2 + xy + yz + zx ≤ $\frac{3}{2}$; d) xy + yz + zx ≤ $\frac{x+y+z}{2}$; e) xy + yz + zx ≤ $\frac{1}{2}$ + 2xyz.

Solution. From 1 = x^2 + y^2 + z^2 +2xyz ≥ $4\sqrt[4]{2\text{a}^3\text{b}^3\text{c}^3}$ ⟺ $\text{x}^3\text{y}^3\text{z}^3 \le \frac{1}{512}$ ⟺ xyz ≤ $\frac{1}{8}$. (2)

From (1) we have xyz = $\frac{1-x^2-y^2-z^2}{2}$ and substituting in the (2) one obtains $\frac{1-x^2-y^2-z^2}{2} \le \frac{1}{8}$

where it appears x^2 + y^2 + z^2 ≥ $\frac{3}{4}$. (3)

To prove the point a). Let f be a polynomial which has roots x, y, z, we have

f(t) =(t−x)(t−y)(t−z) = t^3− (x+y+z)t^2 +(xy+yz+zx)t−xyz . (4)

From (1) we hawe xyz = $\frac{1-x^2-y^2-z^2}{2}$ and substituting in a tie and one obtains

⟺ 2t^3− 2(x + y + x)t^2 + 2(xy+yz+zx)t – 1 + x^2 + y^2 + z^2 = 2(t − a)(t − b)(t − c).

We do t = 1 and note cu S = x + y + z and using the inequality C-B-S one obtains

1 – 2S+ S^2 ≤ $\frac{2}{27}$ $(3 - S)^3$ ⟺ (S + 3)^2(2S – 3) ≤ 0 ⟹ 2S – 3 ≤ 0 ⟹ x + y + z ≤ $\frac{3}{2}$.

Equality is obtained for a = b = c = $\frac{1}{2}$.

To prove the point d).In (4) we do t = $\frac{1}{2}$ and using the inequality C-B-S one obtains:

$\frac{1}{8} - \frac{1}{4}$ (x+y+z) + $\frac{1}{2}$ (xy+yz+zx) − xyz ≤ $\frac{1}{27}\left(\frac{3}{2} - (x + y + z)\right)^3$ ⇔

$-\frac{1}{4}[(x + y + z) - 2(xy + yz + zx)\,] \ \le\ xyz\ -\frac{1}{8}+\frac{1}{27}\,(\frac{3}{2} - (x + y + z))^3$

Taking into account the fact that xyz = $\frac{1-x^2-y^2-z^2}{2}$ şi de $x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}$ we find that

$108[(x + y + z) - 2(xy + yz + zx)\,] \ \geq [2(x + y + z)^2 + 3(x + y + z) + 18][3\text{-}2(x + y + z)]$

How $x + y + z \ \le \frac{3}{2}$ results $(x + y + z) - 2(xy + yz + zx) \geq 0 \Leftrightarrow xy + yz + zx \leq \frac{x + y + z}{2}$

Equality is obtained for a = b = c = $\frac{1}{2}$. To prove the point e).

Because $x^2 + y^2 + z^2 + 2xyz = 1$ results $2xyz = 1 - x^2 - y^2 - z^2$ and then $xy + yz + zx \leq \frac{1}{2} + 2xyz \Leftrightarrow x^2 + y^2 + z^2 + xy + yz + zx \leq \frac{3}{2}$

In (4) we do t = x + y + z = S and one obtains $S^3 - S^2 + (xy + yz + zx)S - xyz = (x+y)(y+z)(z+x)$

$\Leftrightarrow S^3 - S(S^2 - xy - yz - zx) - xyz = (x+y)(y+z)(z+x) \Leftrightarrow S^3 - S(x^2 + y^2 + z^2 + xy + yz + zx) - xyz = (x+y)(y+z)(z+x)$ (5)

Because $x + y \geq 2\sqrt{xy}$, $y + z \geq 2\sqrt{yz}$, $z + x \geq 2\sqrt{zx}$ results $(x+y)(y+z)(z+x) \geq 8xyz$

Then (5) becomes $S^3 - S(x^2 + y^2 + z^2 + xy + yz + zx) - xyz \geq 8xyz$ ⇔

$S(x^2 + y^2 + z^2 + xy + yz + zx) \leq S^3 - 9xyz$. To show that $\frac{S^3 - 9xyz}{S} \leq \frac{3}{2}$ ⇔ $2S^3 - 18xyz - 3S \leq 0$. Taking into account that xyz = $\frac{1-x^2-y^2-z^2}{2}$ and the fact that $3(x^2 + y^2 + z^2) \geq S^2$ inequality becomes $2S^3 - 18(1 - \frac{S^2}{3}) - 3S \leq 0 \Leftrightarrow 2S^3 + 3S^2 - 3S - 9 \leq 0 \Leftrightarrow (S^2 + 3S + 3)(2S - 3) \leq 0$ is true because $x + y + z \ \le \frac{3}{2}$. Results $x^2 + y^2 + z^2 + xy + yz + zx \leq \frac{3}{2} \Leftrightarrow xy + yz + zx \leq \frac{1}{2} + 2xyz$. Points a, b), c)) and e) are a problem proposed by M.Tetiva and the point d)

a problem of O.Purcaru, the short list 2003 din [1].

6. Either a, b, c, d positive numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. To show that $ab + cd + (a+b)(c+d) \leq \frac{5}{4} + 4abcd$.

Solution. We note with S = ab +cd +(a+b)(c+d) and with P = abcd.

Consider the polynomial $f(x) = x^4 - (a + b+c+d)x^3 + Sx^2 - (abc+abd+acd+bcd)x + P =$

$(x - a)(x - b)(x - c)(x - d)$.

We have $|f(it)|^2 = \left| t^4 - i(a+b+c+d)t^3 - St^2 - i(abc+abd+acd+bcd)t + P \right|$ (1)

$But\ |A+iB| \geq A \Leftrightarrow |A+iB|^2 \geq A^2 \Leftrightarrow |A+iB|\overline{|A+iB|} \geq A^2 \Leftrightarrow A^2 + B^2 \geq A^2$.

Equality is obtained for q = 0.

The relationship (1) becomes $|f(it)|^2 \geq | t^4 - St^2 + abcd |^2$ (2)

But $|f(it)|^2 = |f(it)|\overline{|f(it)|} = (it-a)(it-b)(it-c)(it-d)(-it-a)(-it-b)(-it-c)(-it-d) =$

$(t^2+a^2)(t^2+b^2)(t^2+c^2)(t^2+d^2)$. (3)

From (2) and (3) we find $| t^4 - St^2 + abcd |^2 \leq (t^2+a^2)(t^2+b^2)(t^2+c^2)(t^2+d^2)$
(4)

The inequality one obtains environments $(t^2+a^2)(t^2+b^2)(t^2+c^2)(t^2+d^2) \leq \frac{1}{256}(4t^2 + a^2 + b^2 + c^2 + d^2)^4 = \frac{1}{256}(4t^2+1)^4$ and (4) becomes $| t^4 - St^2 + abcd |^2 \leq \frac{1}{256}(4t^2+1)^4$.

In this latter connection we do $t = \frac{1}{2}$ and one obtains

$\left| \frac{1}{16} - \frac{1}{4}S + P \right| \leq \frac{1}{4} \Leftrightarrow \left| S - 4P - \frac{1}{4} \right| \leq 1$ where it appears $S \leq \frac{5}{4} + 4P$. It is observed that the complex of (1) for t = 1/2 is 0. So equality can be achieved. Equality is obtained for $a=b=c=d = \frac{1}{2}$.

In all examples presented dificultați that arise in solving inequalities are sometimes in finding the values of t and intermediate stages must go through them.

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