PROPOSED PROBLEMS

1. Given string $a_1 = 1$, $a_{n+1} = 1 + \frac{n}{a_n}$ for $n \geq 1$. To calculate: $\lim_{n\to\infty}\left(\frac{a_n}{\sqrt{n}}\right)^{\sqrt{n}}$.

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Solution: It is proved by induction that

$$\frac{1+\sqrt{4n-3}}{2} \leq a_n \leq \frac{1+\sqrt{4n+1}}{2} \quad , (\forall)\ n \geq 3 \qquad (1)$$

From (1) achieve

$$\frac{1+\sqrt{4n-3}}{2} - \sqrt{n} \leq a_n - \sqrt{n} \leq \frac{1+\sqrt{4n+1}}{2} - \sqrt{n} \qquad (2)$$

Moving to limit the relation (2) get that:

$$\lim_{n\to\infty}\left(a_n - \sqrt{n}\right) = \frac{1}{2}. \qquad (3)$$

In a similar fashion, we $\lim_{n\to\infty}\frac{a_n}{\sqrt{n}} = 1$ (4)

Note that the limit is of the form 1^∞ . Taking into account the (3) and (4) achieve :

$$\lim_{n\to\infty}\left(\frac{a_n}{\sqrt{n}}\right)^{\sqrt{n}} = \lim_{n\to\infty}\left(1+\frac{a_n-\sqrt{n}}{\sqrt{n}}\right)^{\sqrt{n}} = \lim_{n\to\infty} e^{\lim\limits_{n\to\infty}(a_n-\sqrt{n})} = \sqrt{e}\ .$$

2. It is considered the system: $\begin{cases} x(1+3y^2) = y(y^2+3) \\ y(1+3z^2) = z(z^2+3) \\ z(1+3x^2) = x(x^2+3) \end{cases}$

a) To solve the system in the crowd of integers.
b) To solve the system of complex numbers in the crowd.

Proposed by MARCEL CHIRITA, Bucharest, Romania

Solution. a) The system can be written:

$$x = \frac{y^3+3y}{3y^2+1} \ ; \ y = \frac{z^3+3z}{3z^2+1} \ ; \ z = \frac{x^3+3x}{3x^2+x} \ . \quad (1)$$

Note că $x = y = z = 0$ is the solution.

Either $y \geq 1$ a solution. From (1) we obtain $x - y = \frac{(y-1)^3}{3y^2+1}$. So $x \geq y \Rightarrow x \geq 1$ and similarly we obtain that $z \geq 1 \Rightarrow y \geq z \Rightarrow x \geq y \geq z \geq x \Rightarrow x = y = z = 1$ so $x = y = z = 1$ is the solution.

Either $y \le -1$ a solution. From (1) we obtain $x - y = \frac{(y-1)^3}{3y^2+1}$. So $x \le y \Rightarrow x \le -1$ and similarly we obtain that $z \le -1 \Rightarrow y \le z \Rightarrow x \le y \le z \le x \Rightarrow x = y = z = -1$ so $x = y = z = -1$ is the solution.

Shows solutions (1,1,1) şi (−1, −1, −1).

b) Let (x, y, z) a solution different from those found. Taking into account (1) the system can be written as: $\frac{x+1}{x-1} = \left(\frac{y+1}{y-1}\right)^3$, $\frac{y+1}{y-1} = \left(\frac{z+1}{z-1}\right)^3$, $\frac{z+1}{z-1} = \left(\frac{x+1}{x-1}\right)^3$.

Either the $\varphi : C\text{-}\{1\} \to C$, $\varphi(t) = \frac{t+1}{t-1}$. Then we have $\varphi(x) = \varphi(y)^3$, $\varphi(y) = \varphi(z)^3$, $\varphi(z) = \varphi(x)^3$ and we obtain $\varphi(x)^{27} = \varphi(x) \Rightarrow \varphi(x)^{27}[\varphi(x)^{26} - 1] = 0$. From $\varphi(x) = 0$ we obtain $\frac{x+1}{x-1} = 0 \Rightarrow x = -1$, the solution obtained above. From $\varphi(x)^{26} = 1$ We obtain $\varphi(x) \in U_{26}\text{-}\{-1,1\}$ Bearing in mind the conditions. Either $\omega \in U_{26}\text{-}\{-1,1\}$ i.e.
$\omega = \cos\frac{2k\pi}{26} + i\sin\frac{2k\pi}{26} = \cos\frac{k\pi}{13} + i\sin\frac{k\pi}{13}$, $k \in \{1,2,3,..,12,14,\ldots 25\}$. Then in the
$\varphi(x) = \omega \Rightarrow \frac{x+1}{x-1} = \omega \Rightarrow x = -i\,\mathrm{ctg}\frac{k\pi}{13}$, $k \in \{1,2,3,..,12,14,\ldots 25\}$.

Analog in $\varphi(z) = \omega^3$ we obtain $z = -i\,\mathrm{ctg}\frac{3k\pi}{13}$, $k \in \{1,2,3,..,12,14,\ldots 25\}$ and of the $\varphi(y) = \omega^3$ we obtain $y = -i\,\mathrm{ctg}\frac{9k\pi}{13}$, $k \in \{1,2,3,..,12,14,\ldots 25\}$.

So in C are system solutions: $\{(-1,-1,-1),(0,0,0),(1,1,1)\} \cup \{(-i\,\mathrm{ctg}\frac{k\pi}{13}, -i\,\mathrm{ctg}\frac{9k\pi}{13}, -i\,\mathrm{ctg}\frac{3k\pi}{13})\} \mid k \in \{1,2,..,12,14,\ldots 25\}$.

3. Solve the equation:

$$3^{1-x} + 3^{\sqrt{3x-2x^2}} = 4$$

Proposed by Marcel Chiriţă,Bucharest,Romania

Solution. For $3x - 2x^2 \ge 0$ we have $x \in [0, 3/2]$.
Only solutions of the equation is x = o and x = 1.
Let $x \in [1, 3/2]$ therefore: $1-x < 0 \Rightarrow 3^{1-x} < 1$ and $3x - 2x^2 < 1 \Rightarrow 3^{\sqrt{3x-2x^2}} < 3$ we have $3^{1-x} + 3^{\sqrt{3x-2x^2}} < 4$ contradiction.
Let $x \in [1, 3/2]$ theerefore: $1-x < 0 \Rightarrow 3^{1-x} < 1$ şi $3x - 2x^2 < 1 \Rightarrow 3^{\sqrt{3x-2x^2}} < 3$ we have $3^{1-x} + 3^{\sqrt{3x-2x^2}} < 4$ contradiction.
Let $x \in (0,1)$.

a) $3^{1-x}+3^{\sqrt{3x-2x^2}} = 3.3^{-x}+3^{\sqrt{3x-2x^2}} = 3^{-x}+3^{1-x}+3^{1-x}+3^{\sqrt{3x-2x^2}} \geq 4\sqrt[4]{3^{-3x+\sqrt{3x-2x^2}}} > 4$

If $-3x+\sqrt{3x-2x^2} > 0 \Leftrightarrow 3x-2x^2 > 9x^2 \Leftrightarrow 11x^2-3x<0 \Rightarrow x\in(0,3/11)$. (1)

b) $3^{1-x}+3^{\sqrt{3x-2x^2}} = 3^{1-x}+3.3^{\sqrt{3x-2x^2}-1} = 3^{1-x}+3^{\sqrt{3x-2x^2}-1}+3^{\sqrt{3x-2x^2}-1}+3^{\sqrt{3x-2x^2}-1} \geq$

$4\sqrt{3^{1-x+3(\sqrt{3x-2x^2}-1)}} > 4$. If $1-x+3(\sqrt{3x-x^2}-1) > 0 \Leftrightarrow 3\sqrt{3x-2x^2} > 2+x$

$\Leftrightarrow 9(3x-2x^2) > 4+4x+4x^2 \Leftrightarrow 19x^2-23x+4<0 \Leftrightarrow x\in(4/9,1)$. (2)

From (1) and (2) because $(0,3/11)\cap(4/9,1)=(0,1)$ hence of the equation not solutions if $x\in(0,1)$.

We conclude that the solutions of the equation is x = 0 and x = 1.

4. Solve the equation : $(C_n^2+2C_n^4+3C_n^6+...)-(C_n^3+2C_n^5+3C_n^7+...) = n^2$.

Problem proposed by Prof. Marcel Chirita, Bucharest

Solution. We have $(1+1)^n = C_n^0+C_n^1+C_n^2+C_n^3+....=2^n$ și $(1-1)^n = C_n^0-C_n^1+C_n^2-C_n^3+.... = 2^n \Longrightarrow$

$C_n^0+C_n^2+C_n^4+C_n^6+....=2^{n-1}$ și $C_n^1+C_n^3+C_n^5+C_n^7+....=2^{n-1}$

We have $C_n^m = \frac{n!}{m!(n-m)!} = \frac{n}{m}\cdot\frac{(n-1)!}{(m-1)!(n-m)!} = \frac{n}{m}\cdot C_{n-1}^{m-1}$ then

$S_1 = C_n^2+2C_n^4+3C_n^6+... = \frac{n}{2}(C_{n-1}^1+C_{n-1}^3+C_{n-1}^5+C_{n-1}^7+...) = \frac{n}{2}\cdot 2^{n-2}$

As we have $k\,C_n^{2k+1} = \frac{1}{2}(2k+1-1)\,C_n^{2k+1} = \frac{2k+1}{2}C_n^{2k+1} - \frac{1}{2}C_n^{2k+1} = \frac{n}{2}C_{n-1}^{2k} - \frac{1}{2}C_n^{2k+1}$ then

$S_1 = C_n^3+2C_n^5+3C_n^7+... = \frac{n}{2}(C_{n-1}^2+C_{n-1}^4+C_{n-1}^6+...) - \frac{1}{2}(C_n^3+C_n^5+C_n^7+...) = \frac{n}{2}(2^{n-1}-1)$
$-\frac{1}{2}(2^{n-1}-n) = \frac{n}{2}2^{n-1}-2^{n-2} \Longrightarrow S_1 - S_2 = 2^{n-2}$.

The equation becomes $2^{n-2} = n^2 \Leftrightarrow 2^n = 4n^2$. We note that $n = 2^k$ atunci $2^{2^k} = 2^{2k+2}$
$\Longrightarrow$
$2^k = 2k+2$.

We note that k= 1 and k=2 does not check. k = 3 is the solution . To arătăm by induction that if $n \geq 4$ the equation has no solution. For $k \geq 4$ we will show that $2^k > 2k+2$.

For k = 4 we have $2^4 > 4+2$, true. Suppose that $2^k > 2k+2$. Then $2^{k+1} = 2\cdot 2^k > 2(2k+2)$
=
$4k+4 > 2(k+1)+2$.

It follows that k = 3 is the only solution and then n = 8 is the only solution.

5. The sequence $(a_n)_{n\geq1}$ is defined $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + \frac{a_n}{n^2}$ for all $n \geq 1$. To study the convergence of the sequence.

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Solution. We have $a_{n+2} - a_{n+1} = \frac{a_n}{n^2} \geq 0$ for all $n \geq 1 \Longrightarrow$ The sequence is monotone increasing.

To show that it is convergent we show that it is bounded.

We have $a_3 = 2$, $a_4 = 2+\frac{1}{4}$, $a_5 = 2+\frac{1}{4}+\frac{2}{9} < 3$,...

Be the sequence $x_n = \frac{1}{a_n} - \frac{1}{n-2}$, $n \geq 3$.

Then $\frac{1}{a_n} - \frac{1}{a_{n+1}} = \frac{a_{n+1}-a_n}{a_{n+1}a_n} = \frac{a_{n-1}}{(n-1)^2a_{n+1}a_n} < \frac{1}{(n-1)^2} < \frac{1}{(n-1)(n-2)} = \frac{1}{n-2} - \frac{1}{n-1} \Longrightarrow$

$\frac{1}{a_n} - \frac{1}{a_{n+1}} < \frac{1}{n-2} - \frac{1}{n-1} \Rightarrow \frac{1}{a_n} - \frac{1}{n-2} < \frac{1}{a_{n+1}} - \frac{1}{n-1}$ whence $x_n < x_{n+1}$ $\forall$ n $\geq$3.
Therefore $x_n \geq x_5$ $\forall$ n $\geq$5. $\Rightarrow \frac{1}{a_n} - \frac{1}{n-2} \geq \frac{1}{a_5} - \frac{1}{3} \Rightarrow \frac{1}{a_n} > \frac{1}{a_n} - \frac{1}{n-2} \geq \frac{3-a_5}{3a_5} > 0$.
Results $a_n < \frac{3a_5}{3-a_5}$ $\forall$ n $\geq$5 , where we are that the sequence is bounded. In the end, it appears that the sequence is convergent.

6. Find the maximum value of $\lambda \in R$, so that for any triangle with side lenghts a,b,c, the following inequality holds:

$$2(a^3 + b^3 + c^3) + 3(\lambda + 1)abc \geq (\lambda - 1)(a + b + c)(ab + bc + ca)$$

Proposed by Marcel Chirita,Bucharest,Romania

Solution. If a =1 , b=1 ,c =1 therefore $6 + 3(1+\lambda) \geq 9(\lambda-1) \Rightarrow 9 + 3\lambda \geq 9\lambda - 9 \Rightarrow \lambda \leq 3$.
For $\lambda = 3$ the inequality becoms $2(a^3 + b^3 + c^3) + 12abc \geq 2(a+b+c)(ab+bc+ca) \Leftrightarrow$
$2(a^3 + b^3 + c^3) + 12abc - 2(a+b+c)(ab+bc+ca) \geq 0 \Leftrightarrow$
$a^3 + b^3 + c^3 - 3abc + 9abc - (a^2b + a^2c + b^2a + b^2c + c^2a + c^2b + 3abc) \geq 0 \Leftrightarrow$
$(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) - [(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - 6abc] \geq 0 \Leftrightarrow$
$(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) - [a(b-c)^2 + b(c-a)^2 + c(a-b)^2] \geq 0 \Leftrightarrow$
$2(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) - 2[a(b-c)^2 + b(c-a)^2 + c(a-b)^2] \geq 0 \Leftrightarrow$
$(a+b+c)[(b-c)^2 + (c-a)^2 + (a-b)^2] - 2[a(b-c)^2 + b(c-a)^2 + c(a-b)^2] \geq 0 \Leftrightarrow$
$(b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 + (a+b-c)(a-b)^2 \geq 0$,true.
Maximum value of $\lambda = 3$

7. Determinthe real numbers $a_1, a_2, \dots, a_n$ for which we have:
$\max(a_1, a_2 + a_3 + a_4 + \dots + a_n) = \max(a_2, a_1 + a_3 + a_4 + \dots + a_n) = \max(a_3, a_1 + a_2 + a_4 + \dots + a_n) = \dots = \max(a_n, a_1 + a_2 + a_3 + \dots + a_{n-1}) = 1$.

Proposed by Marcel Chirita,Bucharest,Romania

Solution.
If n = 1, then $a_1 = 1$ . . If n = 2, then $a_1 = a_2 = 1$. Let n $\geq$ 3.
We consider $S = a_1 + a_2 + a_3 + a_4 + \dots + a_n$. The given relation becomes

$$\max(a_1, S - a_1) = \max(a_2, S - a_2) = \max(a_3, S - a_3) = \dots = \max(a_n, S - a_n) = 1.$$

If one i exist for who $a_i > 1$ the given relation is not satisfied. So the whole numbers a_i are smaller than 1 or equal to 1.
We have to analyse the following cases:

1) If $a_1 = a_2 = a_3 = \dots = a_n = 1$ results max (1,n-1) $\neq 1$,contradiction.
2) If n – 1 numbers are equal 1, suppose for example $a_1 = a_2 = a_3 = \dots = a_{n-1} = 1$ and $a_n < 1$.Results $\max(a_n, S - a_n) > 1$, contradiction.
3) If k , $0 \leq k \leq n - 2$, the numbers are equal to 1 than we have the case: $a_1 = a_2 = \dots = a_k = 1$

From the given relation results $S - a_{k+1} = 1, S - a_{k+2} = 1, \dots, S - a_n = 1$. (1)
Making the sum of these n – k relations we obtain : $(n-k)S - (a_{k+1} + a_{k+2} + \dots + a_n) = n - k$
But $a_{k+1} + a_{k+2} + \dots + a_n = S - k$ results $(n-k)S - (S-k) = n - k$ so $S = \frac{n-2k}{n-k-1}$ with the condition $n-k-1 \neq 0$ and from the system (1) we find that

$$a_{k+1} = a_{k+2} = \dots = a_n = S - 1 = \frac{1-k}{n-k-1}.$$

So the solution is $\left(1,1,1\dots 1, \frac{1-k}{n-k-1}, \frac{1-k}{n-k-1}, \dots, \frac{1-k}{n-k-1}\right)$, 1- k times , $\frac{1-k}{n-k-1}$ – n-k times $\forall$ k = 0,1,2,…. ,n-2 . For k=1 none of the numbers is equal to 1.

Finally we have C_n^k solutions, on k places we have 1 and on the others n-k places left have $\frac{1-k}{n-k-1}$, k = 0,1,2,…n-2.
We have $C_n^0 + C_n^1 + \ldots + C_n^{n-2} = 2^n - C_n^{n-1} - C_n^n = 2^n - n - 1$ solutions.
*Proposed Determinthe real numbers $a_1, a_2, \ldots, a_n$, $n \geq 3$ for which we have:
$\max(a_1^2, a_2 + a_3 + a_4 + \ldots + a_n) = \max(a_2^2, a_1 + a_3 + a_4 + \ldots + a_n) = \max(a_3^2, a_1 + a_2 + a_4 + \ldots + a_n) = \ldots = \max(a_n^2, a_1 + a_2 + a_3 + \ldots + a_{n-1}) = 1$.

8. Let ABC be an triangle . To determine the $\lambda \in R$ so:

$$\left(\frac{S}{R}\right)^2 \leq \lambda \left(\sum \frac{ab\sqrt{ab}}{a+b}\right) \leq \left(\frac{S}{2r}\right).$$

Problem proposed by Prof. Marcel Chirita, Bucharest

Solution.Let a=b=c=1 , then $S = \frac{\sqrt{3}}{4}$, $R=\frac{\sqrt{3}}{3}$, $r=\frac{\sqrt{3}}{6}$ and one obtains $\frac{9}{16} \leq \frac{3\lambda}{2} \leq \frac{9}{16}$ where $\lambda = \frac{3}{8}$.
Must show that $\sum \frac{ab\sqrt{ab}}{a+b} \leq \frac{8}{3}\left(\frac{S}{2r}\right)^2 \Leftrightarrow \sum \frac{ab\sqrt{ab}}{a+b} \leq \frac{8}{3}\left(\frac{pr}{2r}\right)^2 = \frac{2p^2}{3}$.
Taking into account the $\sqrt{ab} \leq \frac{a+b}{2}$ and recovery is enough to show that $\sum \frac{bc}{2} \leq \frac{2p^2}{3}$
$\Leftrightarrow 3(ab + bc + ca) \leq (a+b+c)^2 \Leftrightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$, evident.
To demonstrate now $\sum \frac{ab\sqrt{ab}}{a+b} \geq \frac{8}{3}\left(\frac{S}{R}\right)^2$.
Taking into account the $\sqrt{ab} \geq \frac{2ab}{a+b}$ and recovery is enough to show that $\Leftrightarrow$
$\sum \frac{a^2b^2}{(a+b)^2} \geq \frac{4}{3}\left(\frac{S}{R}\right)^2$.
The inequality C − B − S we have $\sum \frac{a^2b^2}{(a+b)^2} \sum 1 \geq \left(\sum \frac{bc}{b+c}\right)^2$ and it is enough to show that
$\left(\sum \frac{bc}{b+c}\right)^2 \geq 4\left(\frac{S}{R}\right)^2 \Leftrightarrow \sum \frac{ab}{a+b} \geq \frac{2S}{R} \Leftrightarrow \sum \frac{ab}{2S(a+b)} \geq \frac{1}{R} \Leftrightarrow \sum \frac{1}{(a+b)\sin C} \geq \frac{1}{R} \Leftrightarrow \sum \frac{2R}{(a+b)c} \geq \frac{1}{R}$
$\Leftrightarrow \sum \frac{1}{(a+b)c} \geq \frac{1}{2R^2}$
The inequality C-B-S we have $\sum \frac{1}{(a+b)c} \cdot \sum (a+b)c \geq 9$ results $\sum \frac{1}{(a+b)c} \geq \frac{9}{\sum(a+b)c} = \frac{9}{2(ab+bc+ca)}$. In view of Leibniz's inequality we have:
$ab + bc + ca \leq a^2 + b^2 + c^2 \leq 9R^2$ one obtains that $\sum \frac{1}{(a+b)c} \geq \frac{1}{2R^2}$.

9. Solve in real numbers the system:

$$2^x + 2^y = 12$$
$$3^x + 3^y = 36$$

Propose by Marcel Chiriţă,Bucharest,Romania

Solution: We introducem the notation $x = 2 + a$ şi cu $y = 3 + b$ we obtain the system:

$$2^a + 2 . 2^b = 3$$
$$3^a + 3 . 3^b = 4$$

From $2^a = 3 - 2^{b+1} \geq 0 \Rightarrow b + 1 \leq \log_2 3 \Rightarrow b \leq \log_2 3 - 1$.
From $3^a = 4 - 3^{b+1} \geq 0 \Rightarrow b + 1 \leq \log_3 4 \Rightarrow b \leq \log_3 4 - 1$.
Hence $b \in (-\infty, \log_3 4 - 1)$ because $\log_2 3 - 1 > \log_3 4 - 1$.
From $2^x + 2^y = 12$ obtain $a = \log_2(3 - 2^{b+1})$ again from $3^x + 3^y = 36$
obtain = $\log_3(4 - 3^{b+1})$.

Consider function f: (- ∞ , $\log_3 4 - 1$) → R ,

$$f(x) = \log_2 (3 - 2^{x+1}) - \log_3 (4 - 3^{x+1}) .$$

Derivative function ist $f'(x) = \frac{-2^{x+1}}{(3-2^{x+1})} - \frac{-3^{x+1}}{(4-3^{x+1})} = \frac{-2^{x+1}(4-3^{x+1}) + 3^{x+1}(3-2^{x+1})}{(3-2^{x+1})(4-3^{x+1})}$

From $f'(x) = 0$ obtain $-8 \cdot 2^x + 6 \cdot 6^x + 9 \cdot 3^x - 6 \cdot 6^x = 0 \Rightarrow -8 \cdot 2^x + 9 \cdot 3^x = 0 \Rightarrow$

$9 \cdot 3^x = 8 \cdot 2^x \Rightarrow \left(\frac{3}{2}\right)^x = \left(\frac{8}{9}\right) \Rightarrow x_o = \frac{\ln 8 - \ln 9}{\ln 3 - \ln 2}$, alone solution.

Obtain:

x	-∞	x_o	$\log_3 4$
f′	-----------	0	+++++++
f	+∞	$f(x_o)$	+∞

Then f(x) is decreasing on $(-\infty, x_o)$ and increasing on (x_o, ∞) , with a minimum at $x_o = \frac{\ln 8 - \ln 9}{\ln 3 - \ln 2}$.

If $f(x_o) < 0$ therifore equation $f(x) = 0$ have maximum two solutions

Observam that(a=0,b=0) şi (a=1 ,b= -1) have alone solutions.

Resulting that the system have in the (x=2,y=3) şi (x=3,y=2) alone solutions.