A plane extension of the symmetry relation

Alexandru Blaga
National College "Mihai Eminescu"
Satu Mare, Romania
alblga2005@yahoo.com

The idea of this article comes as a response to a problem of "symmetry". The defining of this new symmetry leads in particular cases at the symmetry to a line or to a point, in the plane. More exactly, considering the "symmetric" of a point P_0 from the sides of a triangle, to a line which includes one of the triangle's vertices,we obtain extensions of a classic problem.

Let P_0 be a point on the side $[BC]$ and a line d, $C \in d$ and $d \cap AB \neq \emptyset$. The symmetric of P_0 is defined like this:

a) $\mathcal{S}(P_0) = P_0'$, $P_0' \in [AC]$;

b) $[P_0'P_0] \cap d = \{C_1\}$;

c) $[P_0C_1] \equiv [C_1P_0']$ (figure 1).

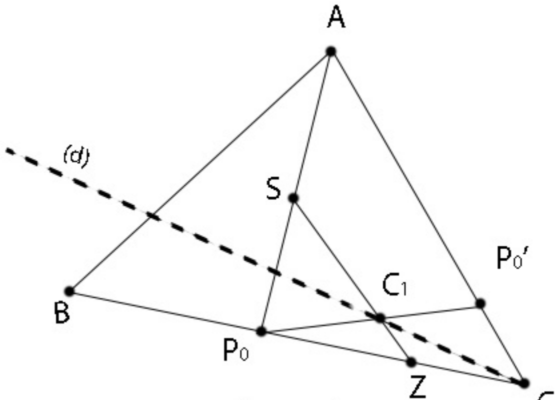

Figure 1.

This kind of construction is possible in two ways:

1. If S is the midpoint of the segment $[AP_0]$ and Z the midpoint of the segment $[P_0C]$, so $[SZ] \cap d = \{C_1\}$ and $SZ \parallel AC$ and the mid-segment in the triangle ABC. Hence, C_1 is the midpoint of the segment $[P_0P_0']$.

2. Let $P_0M \parallel AC$, $M \in d$ and $MK \parallel BC$, $K \in AC$. We have P_0MKC a parallelogram, which means that the diagonals are halved, $[P_0K] \cap [MC] = \{C_1\}$. So, from this construction we can deduce the terms in which this "symmetric" can be considered: in the first construction, $d \cap (SP_0) \neq \emptyset$ and in the second one M must be a point inside the triangle.

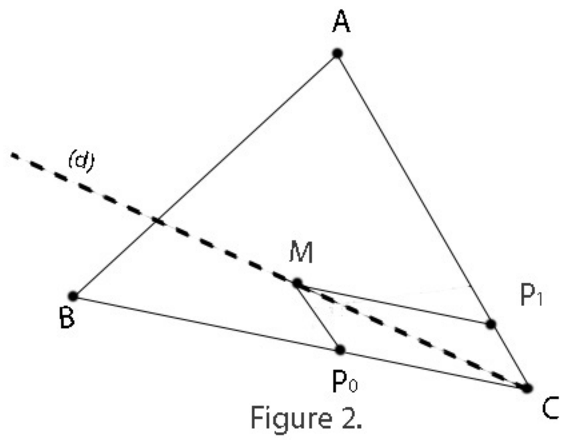

Figure 2.

In every situation that we are going to approach, the position of the points whose "symmetric" we will consider, they will meet one of this conditions. It can be proved that they are equivalent. If line d is the bisector of the angle , then the defined symmetric is that one from the classic acceptance.

It is necessary that $P_0' \in (AC)$. Based on the sense defined by this symmetry at a), b) and c), we will note the symmetric of P_0 with $\mathcal{S}(P_0)$.

We are interested in the following problem: Considering a point $P_0 \in [BC]$ and $\mathcal{S}(P_0) = M$, $M \in [AC]$. For $d_1 \cap BC \neq \emptyset$, $A \in d_1$ and $\mathcal{S}(M) = R$, $R \in [AD]$ and for $d_2 \cap [AC] \neq \emptyset$, $B \in d_2$, $\mathcal{S}(R) = V$, $V \in [BC]$.

Following this described method in this way: in relation to d, in relation to d_1, in relation to d_2 we obtain the points M', R', V' on the order. So we performed this symmetry in the order d, d_1, d_2 six times. The problem that interests us is in which conditions $V' = P_0$, so the point P_0 comes back in the initial position. In support of this property, we consider the article [1].

In the mentioned article, it is proved the fact that after six steps it reaches the starting point, if the lines which the symmetry are related to are angle bisectors. It will be also studied the case in which after three steps the point come back in the initial position and also the concurrence of the lines BM, CR, AV.

Theorem 1. Let AN, CK, BI be three cevians of the triangle ABC and $AN \cap CK \cap BI \neq \emptyset$, where $N \in (BC)$, $K \in (AB)$, $I \in (AC)$. If $P_0 \in (BC)$ and $\frac{CP_0}{CB} = \frac{1}{k}$, $\frac{BK}{KA} = \frac{1}{p_1}$, $\frac{CN}{NB} = \frac{1}{p_2}$ so that

(i) $p_1 > 1 > \frac{1}{k}$;

(ii) $kp_1(1 - p_2) < 1$;

(iii) $k(1 + p_1 - p_1p_2) > 1$

then after six consecutive "symmetries", the point P_0 comes back in P_0.

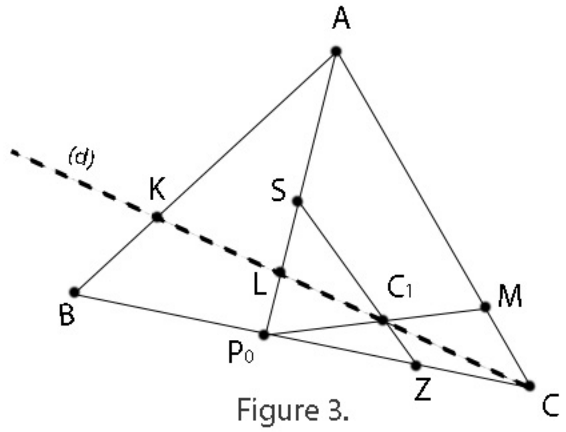

Figure 3.

Proof. We will show the steps by which we make the construction.

Step 1 (figure 3) . We have $[AS] \equiv [SP]$, $[CZ] \equiv [ZP_0]$, $CK \cap [SP_0] = \{L\}$. Using the Menelaus's Theorem in $\triangle ABP_0$ with transversal $C - L - K$, we have $\frac{1}{k} \cdot \frac{1}{p_1} \cdot \frac{AL}{LP_0} = 1$, so $\frac{LP_0}{AL} = \frac{1}{kp_1}$ and $\frac{LP_0}{AP_0} = \frac{1}{1+kp_1}$. We deduce that $SL = SP_0 - LP_0 = \frac{AP_0}{2} - \frac{AP_0}{1+kp_1} = \frac{AP_0}{2} \cdot \frac{kp_1-1}{kp_1+1}$. Using the same theorem, but in $\triangle SP_0Z$ with the transversal $C - C_1 - L$, we obtain $\frac{1}{2} \cdot \frac{LP_0}{SL} \cdot \frac{SC_1}{C_1Z} = 1$, as $\frac{SC_1}{C_1Z} = kp_1 - 1 = \frac{AM}{MC}$ and $\frac{AM}{AC} = \frac{kp_1-1}{kp_1}$ (1) and this fraction exists because $kp_1 > 1$, this condition also assures that $L \in [SP_0]$.

Step 2 (figure 4). We have $[S_1B] \equiv [S_1M]$, $[AU] \equiv [MU]$, $[AN] \cap [BM] = \{V_1\}$.

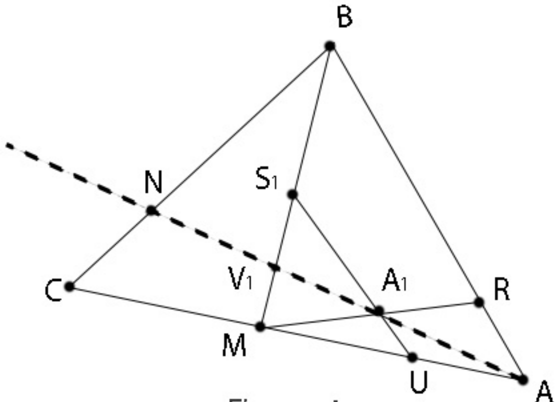

Figure 4.

We have $\frac{AM}{AC} \cdot \frac{CN}{NB} \cdot \frac{BV_1}{V_1M} = 1$, $\frac{kp_1-1}{kp_1} \cdot \frac{1}{p_2} \cdot \frac{BV_1}{V_1M} = 1$, $\frac{V_1M}{BV_1} = \frac{kp_1-1}{kp_1p_2}$ and $\frac{V_1M}{BM} = \frac{kp_1-1}{kp_1p_2+kp_1-1}$. So $S_1V_1 = S_1M - V_1M = \frac{BM}{2} - BM \cdot \frac{kp_1-1}{kp_1p_2} = \frac{BM}{2} \cdot \frac{kp_1p_2-kp_1+1}{kp_1p_2}$. With the same theorem in $\triangle S_1MU$, we deduce that $\frac{1}{2} \cdot \frac{V_1M}{V_1S_1} \cdot \frac{S_1A_1}{A_1U} = 1$ and $\frac{S_1A_1}{A_1U} = \frac{BR}{RA} = \frac{kp_1p_2-kp_1+1}{kp_1-1}$ (2), so that $\frac{BR}{BA} = \frac{kp_1p_2-kp_1+1}{kp_1p_2}$. which exists from the condition (ii).Moreover, from the condition (ii), we have $V_1M < \frac{BM}{2} = SM$.

Step 3 (figure 5). We have $[CW] \equiv [WR]$, $[BJ] \equiv [RJ]$, $[BI] \cap [WR] = \{R_1\}$.

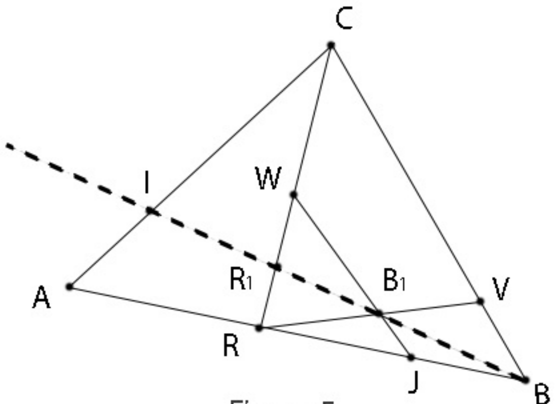

Figure 5.

$\frac{BR}{BA} \cdot \frac{AI}{IC} \cdot \frac{CR_1}{R_1R} = 1$ and because the concurrence $AN \cap CK \cap BI \neq \emptyset$ leads to $\frac{AI}{IC} \cdot \frac{CN}{NB} \cdot \frac{BK}{KA} = 1$, then $\frac{AI}{IC} = p_1p_2$. We can write the relation $\frac{kp_1p_2 - kp_1 + 1}{kp_1p_2} \cdot p_1p_2 \cdot \frac{CR_1}{R_1R} = 1$ and we deduce that $\frac{R_1R}{CR} = \frac{kp_1p_2 - kp_1 + 1}{k + kp_1p_2 - kp_1 + 1}$; $WR_1 = \frac{CR}{2} \cdot \frac{k - kp_1p_2 + kp_1 - 1}{k + kp_1p_2 - kp_1 + 1}$. From the Menelaus's Theorem in $\triangle WRJ$, we obtain $\frac{WB_1}{WJ} = \frac{CV}{CB} = \frac{k + kp_1 - kp_1p_2 - 1}{k}$ (3), which exists from the condition (iii). Moreover, from the condition (iii), we have $R_1R < \frac{CR}{2} = WR$.

We get from the succesive application of the first three steps, the points M, R, U starting from the point $P_0 \in [BC]$.

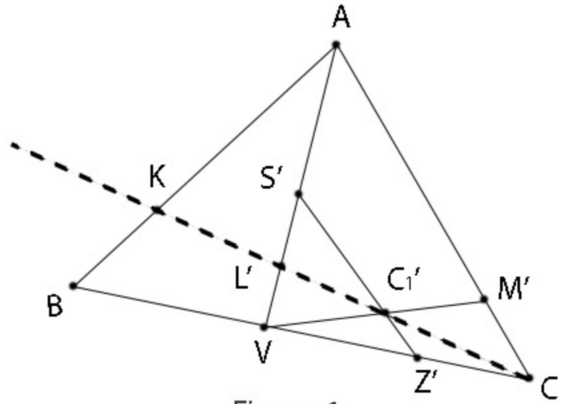

Figure 6.

In the following calculations, we put $\frac{k + kp_1 - kp_1p_2 - 1}{k} = p_3$. Point V can take the place of P_0, because $\frac{CV}{CB} = p_3 < 1$: $k + kp_1 - kp_1p_2 - 1 < k$ leads to $kp_1(1 - p_2) < 1$, so (ii).

Step 4 (figure 6). We have $[S'A] \equiv [S'V]$, $[CZ'] \equiv [Z'V]$, $[CK] \cap [S'V] = \{L'\}$. From the relations $\frac{CV}{CB} \cdot \frac{BK}{AK} \cdot \frac{AL'}{L'V} = 1$ and $\frac{CZ'}{CV} \cdot \frac{VL'}{L'S'} \cdot \frac{S'C_1}{C_1'Z'} = 1$, we conclude that $\frac{L'V}{AL'} = \frac{p_3}{p_1}$, $L'V = AV \cdot \frac{p_3}{1 + p_3}$, $S'L' = \frac{AV}{2} \cdot \frac{p_1 - p_3}{p_1 + p_3}$, $\frac{S'C_1'}{C_1'Z'} = \frac{AM'}{M'C} = \frac{p_1 - p_3}{p_3}$ and finally $\frac{AM'}{AC} = \frac{p_1 - p_3}{p_1}$ (4). The inequality $p_1 > p_3$ is equivalent with $p_1 > \frac{k + kp_1 - kp_1p_2 - 1}{k}$ or $kp_1p_2 + 1 > k$, which is provided from (i) and (ii), because $kp_1p_2 + 1 > kp_1 > k(p_1 - 1) > 0$.

Step 5 (figure 7). We have $[BS_1'] \equiv [S_1'M']$, $[AU'] \equiv [M'U']$, $[AN] \cap [S_1'M'] = \{V_1'\}$.

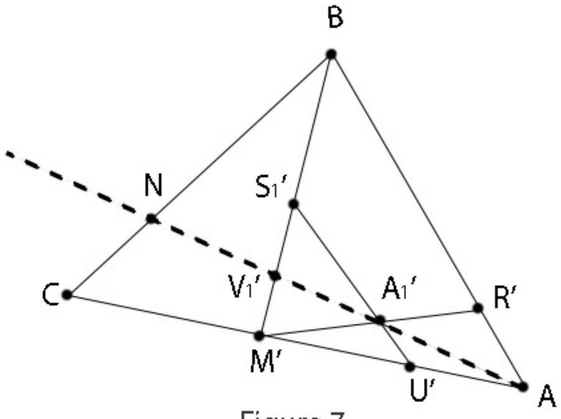

Figure 7.

From the relations $\frac{AM'}{AC}\cdot\frac{CN}{NB}\cdot\frac{BV_1'}{V_1'M'}=1$ and $\frac{AV'}{AM'}\cdot\frac{M'V_1'}{V_1'S_1'}\cdot\frac{S_1'A_1'}{A_1'U'}=1$, we conclude that $\frac{V_1'M'}{BV_1'}=\frac{p_1-p_3}{p_1p_2}$, $\frac{V_1'M'}{BM'}=\frac{p_1-p_3}{p_1-p_3+p_1p_2}$, $S_1'V_1'=\frac{BM'}{2}-V_1'M'=\frac{BM'}{2}\cdot\frac{p_3-p_1+p_1p_2}{p_1-p_3+p_1p_2}$, $\frac{S_1'A_1'}{A_1'U'}=\frac{BR'}{R'A}=\frac{p_3-p_1+p_1p_2}{p_1p_2}$ (5). Condition $p_3-p_1+p_1p_2>0$ is equivalent with $k>1$, which is provided from the condition (i).

Step 6 (figure 8). We have $[CW']\equiv[W'R']$, $[BJ']\equiv[R'J']$, $[W'R']\cap[BI]=\{R_1'\}$.

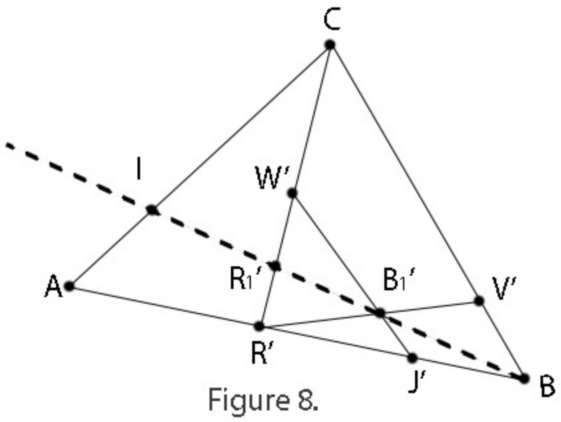

Figure 8.

From the relations $\frac{BR'}{BA}\cdot\frac{AI}{IC}\cdot\frac{CR_1'}{R_1'R'}=1$ and $\frac{BJ'}{BR'}\cdot\frac{R'R_1'}{W'R_1'}\cdot\frac{W_1'B_1'}{B_1'J'}=1$, we conclude that $\frac{R_1'R'}{CR_1'}=\frac{p_3-p_1+p_1p_2}{1}$, because from the concurrence of the lines BI, AN

and CK we have $\frac{AI}{IC} = p_1 p_2$. On the other hand, $\frac{R'R_1'}{CR'} = \frac{p_3 - p_1 + p_1 p_2}{1 + p_3 - p_1 + p_1 p_2}$, $W'R_1' = \frac{CR'}{2} \cdot \frac{1 - p_3 + p_1 - p_1 p_2}{1 + p_3 - p_1 + p_1 p_2}$, so we have $\frac{W'B_1'}{B_1'J'} = \frac{CV'}{V'B} = \frac{1 - p_3 + p_1 - p_1 p_2}{p_3 - p_1 + p_1 p_2}$, then $\frac{CV'}{CB} = \frac{1 - p_3 + p_1 - p_1 p_2}{1}$ (6).

The conclusion $V' = P_0$ goes to $1 - p_3 + p_1 - p_1 p_2 = \frac{1}{k}$ or $1 + p_1 - p_1 p_2 - \left(1 + p_1 - p_1 p_2 - \frac{1}{k}\right) = \frac{1}{k}$, so the theorem is proved.

Theorem 2. In the conditions from the first theorem, the concurrence of the lines BM, CR, AV is equivalent with the condition $P_0 = V$.

Proof. This task actually mean the condition after three succesive "symmetries", the point P_0 comes back in the initial position. From the third step we have $\frac{CV}{CB} = \frac{k + kp_1 - kp_1 p_2 - 1}{k}$ and from the first three steps, we obtain the relations (3), (2) and (1) $\frac{CV}{VB} = \frac{k + kp_1 - kp_1 p_2 - 1}{kp_1 p_2 + 1 - kp_1}$, $\frac{BR}{RA} = \frac{kp_1 p_2 - kp_1 + 1}{kp_1 - 1}$, $\frac{AM}{MC} = \frac{kp_1 - 1}{1}$.The concurrence of the lines BM, CR, AV is equivalent from the Ceva's Theorem, with $\frac{AM}{MC} \cdot \frac{CV}{VB} \cdot \frac{BR}{RV} = 1$, so $k + kp_1 - kp_1 p_2 - 1 = 1$, or $k(1 + p_1 - p_1 p_2) = 2$, so $1 + p_1 - p_1 p_2 = \frac{2}{k}$ (7).

The condition of returning is $\frac{CV}{CB} = \frac{CP_0}{CB}$, so $\frac{k + kp_1 - kp_1 p_2 - 1}{k} = \frac{1}{k}$, or $1 + p_1 - p_1 p_2 = \frac{2}{k}$, so relation (7). In the given conditions from the Theorem 1, the point P_0 is not unique, if the concurrence of cevians are fixed.

We are going to analyze two particular cases, when the cevians AN, BI, CK are medians and angle bisectors, cases were we determine the points which returns in the initial position after three consecutive "symmetries".

(A) Let AN, BI, CK be medians. In this case, $p_1 = p_2 = 1$ and the condition of returning is $\frac{2}{k} = 1$, or $k = 2$, $P_0 = N$, so P_0 is the midpoint of $[BC]$. Because $[P_0 C_1] \equiv [C_1 M]$ and $[BK] \equiv [AK]$, we deduce that $[P_0 M]$ is a mid-segment, namely $M = I$. So the symmetry leads in this case at the medians' feet, so $P_0 = N$.

(B) Let AN, BI, CK be angle bisectors, then we have the usual symmetry and as a result of [1], P_0 is the tangent point of the inscribed circle with the side $[BC]$. It's true the fact that in this case, $p_1 = \frac{b}{a}$, $p_2 = \frac{c}{b}$, $p_1 p_2 = \frac{c}{a}$ and $1 + p_1 - p_1 p_2 = \frac{2}{k}$ is equivalent with $\frac{a+b-c}{2a} = \frac{CP_0}{CB} = \frac{CP_0}{a}$, so $CP_0 = \frac{a+b-c}{2}$. This is the segment determined by point C and the tangent point of the inscribed circle with the side $[BC]$, where $a = BC$, $b = AC$ and $c = AB$.

Annotation. Theorem 1 ensures enough conditions for the returning of the point P_0 after six steps in the initial position. We are going to prove that in more restrictive conditions, the condition of concurrence of the cevians AN, BI, CK is also required. We will note $\frac{BI}{IC} = \alpha \in \mathbb{R}$. It is obvious the fact that if after three consecutive symmetries returns there. The case in which after three steps the point P_0 returns back and this think is possible if $\alpha \cdot \frac{kp_1p_2-kp_1+1}{kp_1p_2} = 1 - \frac{1}{k}$, doesn't show interest. Following the previous process, we will exclude this case.

Theorem 3. Let $P_0 \in [BC]$ and AN, BI, CK three cevians for which $\frac{CP_0}{CB} = \frac{1}{k}$, $\frac{BK}{KA} = \frac{1}{p_1}$, $\frac{CN}{NB} = \frac{1}{p_2}$ such that:

(i) after six consecutive "symmetries", the point P_0 returns in the initial position;

(ii) $p_1 > 1 > \frac{1}{k}$;

(iii) $kp_1(1-p_2) < 1$;

(iv) $\alpha < \frac{BA}{BR}$;

(v) $\alpha < \frac{BA}{BR} \cdot (1 - p_1 + p_1p_2)$.

Then $\alpha = p_1p_2$, so that the cevians are concurrent.

Proof. The ratio $\frac{BI}{IC} = \alpha$ is considered just at the third step, so we are going to retake the reasoning from there.

Step 3′. We have the relations (figure 5): $\frac{kp_1p_2-kp_1+1}{kp_1p_2} \cdot \alpha \cdot \frac{CR_1}{R_1R} = 1$, $\frac{BJ}{BR} \cdot \frac{R_1R}{WR_1} \cdot \frac{WB_1}{B_1J} = 1$, so that $\frac{WB_1}{B_1J} = \frac{kp_1p_2-\alpha(kp_1p_2-kp_1+1)}{\alpha(kp_1p_2-kp_1+1)} = \frac{CV}{VB}$, $\frac{CV}{CB} = \frac{kp_1p_2-\alpha(kp_1p_2-kp_1+1)}{kp_1p_2} = p_3'$ (7).

Step 4′. We have the relations (figure 6): $\frac{CV}{CB} \cdot \frac{1}{p_1} \cdot \frac{AL'}{L'V} = 1$, $\frac{CZ'}{CV} \cdot \frac{L'V}{L'S'} \cdot \frac{S'C_1'}{C_1'Z'} = 1$, $\frac{p_3'}{p_1} \cdot \frac{AL'}{L'V} = 1$, $\frac{S'C_1'}{C_1'Z'} = \frac{p_1-p_3'}{p_3'} = \frac{AM'}{M'C} = 1$, $\frac{AM'}{AC} = \frac{p_1-p_3'}{p_1}$ (8).

Step 5′. We have the relations (figure 7): $\frac{AM'}{AC} \cdot \frac{CN}{NB} \cdot \frac{BV_1'}{V_1'M'} = \frac{AU'}{AM'} \cdot \frac{M'V_1'}{V_1'S_1'} \cdot \frac{S_1'A_1'}{A_1'U'} = 1$, from where we deduce that $V_1'M' = BM' \cdot \frac{p_1-p_3'}{p_1p_2+p_1-p_3}$, $\frac{BR'}{BA} = \frac{p_1p_2+p_3'-p_1}{p_1p_2}$ (9).

Step 6′. We have the relations (figure 8): $\frac{BR'}{RA} \cdot \alpha \cdot \frac{CR_1'}{R_1'R'} = \frac{BJ'}{BR'} \cdot \frac{R'R_1'}{W'R_1'} \cdot \frac{W'B_1'}{B_1'J'} = 1$, from where we deduce that $\frac{R_1'R'}{CR'} \frac{\alpha(p_1p_2+p_3'-p_1)}{p_1p_2+\alpha(p_1p_2+p_3'-p_1}$, $\frac{CV'}{CB} = \frac{p_1p_2-\alpha(p_1p_2+p_3'-p_1)}{p_1p_2}$ (10). The condition of returning is $\frac{CV'}{CB} = \frac{CP_0}{CB}$, where from $\frac{p_1p_2-\alpha(p_1p_2+p_3'-p_1)}{p_1p_2} =$

$\frac{1}{k}$. Transforming the last relation after the replacing of p_3', we obtain $p_1p_2 - \alpha p_1p_2 + \alpha p_1 - \alpha(1-\alpha+\frac{\alpha}{p_2}-\frac{\alpha}{kp_1p_2}) = \frac{p_1p_2}{k}$, $(p_1p_2-\alpha)+\alpha(\alpha-p_1p_2)+\frac{\alpha}{p_2}(p_1p_2-\alpha)+\frac{1}{kp_1p_2}(\alpha-p_1p_2)(\alpha+p_1p_2)=0$. After remaning the common factor, we have $(\alpha-p_1p_2)\left(\alpha-1-\frac{\alpha}{p_2}+\frac{\alpha+p_1p_2}{kp_1p_2}\right)=0$. The case $\alpha-1-\frac{\alpha}{p_2}+\frac{\alpha+p_1p_2}{kp_1p_2}=0$ leads to $\alpha\left(kp_1p_2-kp_1+1\right)=1-\frac{1}{k}$, $\frac{1}{k}=\frac{CV}{CB}=\frac{CP_0}{CB}$, so $V=P_0$. This was removed, so $\alpha=p_1p_2$, which proves the required concurrence.

Comment. The condition (iv) ensures relation (7) and the condition (v) ensures relation (10).

References

[1] J. Berglund, R. Taylor, Bisections and reflections *Math. Mag.* **87**(2014), 284-290.

[2] J. Berglund, Creating a full mathematical experience in the classroom, *JOMA*, 2005.

[3] Z. Usiskin, J. Griffin, D. Witonsky, E. Wilmore, The classification of quadrilaters: a study in definition, *Information Age Pub. Charlotte, NC*, 2008.

Summary We reconsidered the notion of symmetry of a point conditioned by two lines. The symmetries defined like this are applied succesively to three cevians. Considering a given point and performing six succesive symmetries, the point returns in the initial position if the three cevians are concurrent. Under certain conditions, the reverse is true.