Considerations regarding the calculus of the limit of a recursive string

For starters we will consider a string defined by the relation:

$$X_{n+1} = \frac{x_n}{n} + \frac{n+1}{n^2} \qquad \text{with } n \geq 1 \text{ and } x_1 = 1$$

If using this string we form another recursive string, namely $Y_n = nX_n$, we shall study which the limit is for this recursive string when "n" tends to ∞.

First, we will try and form an idea regarding the evolution of this string.
Hence:

$$Y_n = nX_n \rightarrow x_n = \frac{Y_n}{n}$$

$$Y_{n+1} = (n+1)X_{n+1}$$

$$X_{n+1} = \frac{x_n}{n} + \frac{n+1}{n^2}, \text{ and}$$

$$Y_{n+1} = (n+1)\left(\frac{x_n}{n} + \frac{n+1}{n^2}\right) = \frac{n+1}{n}x_n + \frac{(n+1)^2}{n^2}$$

$$(1) \quad Y_{n+1} = \left(\frac{n+1}{n^2}\right)Y_n + \left(\frac{n+1}{n}\right)^2$$

This above is the expression of the recursive string Y_n
How does this string evolve ?
Now, we will calculate some terms from the string, namely:

$Y_1 = 1$; $Y_2 = 6$; $Y_3 = 6{,}75$; $Y_4 = 4{,}77$; $Y_5 = 3{,}05$; $Y_6 = 2{,}17$ ....

What do we observe ? That the sequence Y_n is decreasing when $n \geqslant 3$

Let's extend this conclusion for the whole sequence, namely: if we have a regular sequence, L being the limit of this sequence when n tends to infinity, then by admiting the sequence is decreasing we observe that any member of the sequence will be larger than the value of this limit.

In other words, this relation exists:

$$(2) \qquad Y_n > \frac{(n+1)^2}{n^2-n-1}, \quad \forall\, n \geq 3$$

for Y_1, Y_2, Y_3, Y_4, Y_5 and Y_6

Let's prove by using mathematical induction this inequality. We consider level one as the correspondent level for n=3. Let's prove the relation at this level.
Indeed for n=3 we have $Y_3 = 6{,}75 > \frac{(3+1)^2}{9-3-1} = \frac{16}{5} = 3{,}2$ which is true.
We assume the relation as being consistent for the level k and we want to prove it for the level K-1.
From the relations (1) and (2) results:

$$Y_{K+1} = \left(\frac{K+1}{K^2}\right) Y_K + \frac{(K+1)^2}{K^2} > \left(\frac{K+1}{K^2}\right)\left(\frac{(K+1)^2}{K^2-K-1}\right) + \frac{(K+1)^2}{K^2}$$

Because we considered that $Y_k > \frac{(k+1)^2}{k^2-k-1}$;
according to relation (2)
The expression

$$\left(\frac{k+1}{k^2}\right)\left[\frac{(k+1)^2}{k^2-k-1}\right] + \left(\frac{k+1}{k}\right)^2 > \frac{(k+1)^2}{k^2-k-1}$$

But

$$\left(\frac{(k+1)^2}{k^2-k-1}\right) > \frac{(k+2)^2}{(k+1)^2-(k+1)-1}$$

$$\frac{k^2+2k+1}{k^2-k-1} > \frac{k^2+4k+4}{k^2+k-1}$$

We add up k+1 at the numerator:

$$\frac{k^2+2k+1+k+1}{k^2-k-1} > \frac{k^2+4k+4+k+1}{k^2+k-1}$$

$$\frac{3k+2}{k^2-k-1} > \frac{5k+5}{k^2+k-1} > \frac{3k+5}{k^2+k-1}$$

Because 5k+5>3k+5

$$\frac{3k+2}{k^2-k-1} - \frac{3k+5}{k^2+k-1} > 0$$

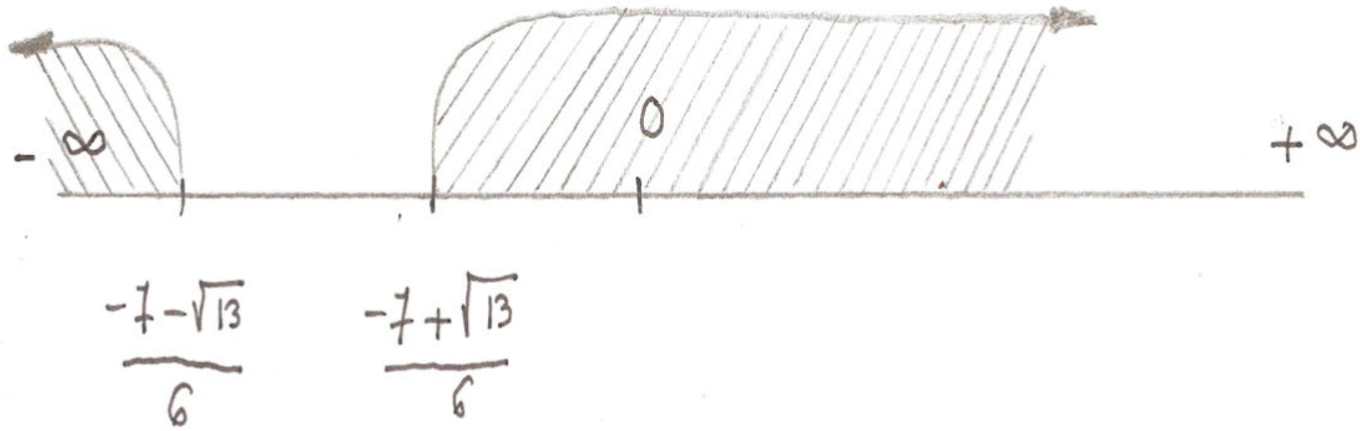
$$(3k+2)(k^2+k-1) - (3k+5)(k^2-k-1) > 0$$

$$3k^2 + 3k^2 - 3k + 2k^2 + 2k - 2 - 3k^2 + 3k^2 + 3k - 5k^2 + 5k + 5 > 0$$

or $\quad 3k^2 + 7k + 3 > 0$

$$\Delta = b^2 - 4ac = 49 - 4\cdot 3\cdot 3 = 49 - 36 = 13 > 0$$

$$k_{1,2} = \frac{-7 \pm \sqrt{13}}{6} \quad \begin{matrix} \nearrow \dfrac{-7+\sqrt{13}}{6} \\ \searrow \dfrac{-7-\sqrt{13}}{6} \end{matrix}$$



For $k > 0$.-

Resulting that relation (2) is valid.
Using again relations (1) and (2) we have:

$$Y_{n+1} - Y_n = \left(\frac{n+1}{n^2}\right) Y_n + \left(\frac{n+1}{n}\right)^2 - Y_n = \left(\frac{n+1}{n}\right)^2 +$$

$$+ \left(\frac{n+1}{n^2} - 1\right) Y_n = \left(\frac{n+1}{n}\right)^2 + \left(\frac{n+1-n^2}{n^2}\right) Y_n =$$

$$= \left(\frac{n+1}{n}\right)^2 + \left(-1 + \frac{n+1}{n^2}\right) Y_n = \left(\frac{n+1}{n}\right)^2 - \left(1 - \frac{n+1}{n^2}\right) Y_n <$$

$$< \frac{(n+1)^2}{n^2} - \left(\frac{n^2-n-1}{n^2}\right)\left(\frac{(n+1)^2}{n^2-n-1}\right) = \text{zero}$$

This concludes that the sequence Y_n is strictly decreasing.
Given that Y_n is regular we calculate its limit.
For that purpose if we consider L as the limit of this sequence, from relation (1) it results that:

$$L = \frac{n+1}{n^2} L + \left(\frac{n+1}{n}\right)^2$$

$$L\left(1 - \frac{n+1}{n^2}\right) = \left(\frac{n+1}{n}\right)^2$$

$$L\left(\frac{n^2-n-1}{n^2}\right) = \frac{n^2+2n+1}{n^2}$$

$$L = \frac{n^2+2n+1}{n^2-n-1} \longrightarrow \lim_{n\to\infty} \frac{n^2+2n+1}{n^2-n-1} = 1 = L$$

We observe that an element of originality like the value of the limit L of the recursive sequence has been deduced even if its expression depends on n.

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