METHOD OF DETERMINATION OF PARAMETERS IN THE CASE OF IRREDUCIBLE FRACTIONS

This article discusses the determination of parameters where a certain fraction is irreducible.
Thus, if we consider a fraction written as P(x)/Q(x), this fraction is irreducible if by decomposing the numerator and the denominator in first-degree polynomials, these polynomials cannot be simplified, not finding any identical ones among them.
We know the decomposition method of the numerator or (and) the denominator by Bezout's theorem.
Thus, if we consider an algebraic equation $P(x) = a_n x^n + a_{n-1} x^{n-1} + \_\ \_ + a_1 x + a_0$
with integer coefficients, this equation admits rational roots of the form p/q, where p is chosen from a set of integer divisors of the free term a_0 and q is chosen from the set of integer divisors of the coefficient a_n of x_n.
In other words, the roots of real numbers of the algebraic equation
$P(x) = a_n x^n + a_{n-1}\ x^{n-1}\ \_\ \_\ \_ + a_{1x} + a_0$ can be found in the set {p/q}.
A similar method can be used as well for the algebraic polynomial Q(x).
Also, if x_1, x_2, x_3 _ _ _ x_n are roots of P(x) and y_1, y_2, y_3 _ _ _ y_n roots of Q(x), then we can write the fraction P(x)/Q(x) as:

$$\frac{P(x)}{Q(x)} = \frac{a_n\,(x - x_1)\,(x - x_2)\,(x - x_3)\ \_\ \_\ \_\ (x - x_n)}{b_n\,(x - y_1)\,(x - y_2)\,(x - y_3)\ \_\ \_\ \_\ (x - y_n)}$$

For this fraction to be simplified, at least one relation of the form $x_k = y_k$ must occur.
This means that we admit at least one root of P(x) to be the root of Q(x) too, in which situation the fraction P(x)/Q(x) is as follows:

$$\frac{P(x)}{Q(x)} = \frac{a_n\,(x - x_1)\,(x - x_2)\ \_\ \_\ \_\ (x - x_k)\ \_\ \_\ \_\ (x - x_n)}{b_n\,(x - y_1)\,(x - y_2)\ \_\ \_\ \_\ (x - x_k)\ \_\ \_\ \_\ (x - y_n)}$$

when the fraction can be simplified by the first-degree polynomial $(x - x_k)$.

In order to verify whether the fraction is reducible or not, it is sufficient to introduce in the expression of Q(x), in turn, the roots x_1, x_2, x_3 _ _ _ x_n of P(x) forming the polynomials Q(x1), Q(x2), and, if in this chain of polynomials we identify a polynomial (or more) for which Q(xk) = 0, this means that the fraction P(x)/Q(x) can be simplified to (x-xk). If the polynomial Q(x) depends on a parameter, let's say "a", then the value of this parameter can be deducted from the relation Q(xk) = 0, as the fraction is required to be reducible or not after the first-degree polynomial (x - xk). Of course that if we identify more polynomials of the form Q(xk) = 0, for example, and Q(x1) = 0, the fraction is simplified twice with first-degree polynomials (x - xk) and (x - xl).
In this situation, we find more options for the parameter "a" which belongs to the set A = {ak, a1} as disjunctive events and this parameter that corresponds to each case of simplification of the fraction P(x)/Q(x).

APPLICATION:

Let is be the fraction $\dfrac{2x^4 + 5x^3 - x^2 + 5x - 3}{x^3 + 6x^2 + 11x - 4a}$

Determine the parameter a so that the fraction is irreducible.

$$\frac{P(x)}{Q(x)} = \frac{2x^4 + 5x^3 - x^2 + 5x - 3}{x^3 + 6x^2 + 11x - 4a}$$

If for the numerator P(x) we use p for the set of integer divisors of the free term, i.e. of (-3), this means that p belongs to the set {±1, ±3} and we also use q for the set of integer divisors of the coefficient of x^4, namely two, i.e. q belongs to the set {±1, ±2}, then the real roots of P(x) will be found in the set f belongs to the set{ $\pm 1, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm 3$}

We find that

$P(1/2) = 2(1/2)^4 + 5(1/2)^3 - (1/2)^2 + 5(1/2) - 3 =$ zero

so it results that 1/2 is a root for P(x).

So the polynomial P(x) can be written as:

$f(x) = (ax^3 + bx^2 + cx + d)(x - 1/2)$

=1/2(ax^3 + bx^2 + cx + d)(2x – 1) = 1/2(2ax^4 + 2bx^3 + 2cx^2 + 2dx – ax^3 – bx^2 – cx – d) = 1/2(2ax^4 + (2b – a)x^3 + (2c – b)x^2 + (2d – c)x – d).

We determine the coefficients a, b, c, d by identification as follows:

(2x^4 + 5x^3 – x^2 + 5x -3) = 1/2(2ax^4 + (2b – a)x^3 + (2c – b)x^2 + (2d – c)x – d)

(2x^4 + 5x^3 – x^2 + 5x -3) = 1/2(2ax^4 + (2b – a)x^3 + (2c – b)x^2 + (2d – c)x – d)

$$(2x^4 + 5x^3 - x^2 + 5x - 3) = ax^4 + \frac{2b-a}{2}x^3 + \frac{2c-b}{2}x^2 + \frac{2d-c}{2}x - \frac{d}{2}$$

of which it results: a = 2

$$\frac{2b-a}{2} = 5$$

$$\frac{2c-b}{2} = -1$$

$$\frac{2d-c}{2} = 5$$

$$-\frac{d}{2} = -3$$

By solving this system compared with a, b, c, d we find that:

$a = 2 \quad b = 6 \quad c = 2 \quad d = 6$, and so we can write that:

$P(x) = 1/2(ax^3 + bx^2 + cx + d)(2x - 1) = 1/2(2x^3 + 6x^2 + 2x + 6)(2x - 1)$ or by simplifying by two $P(x) = (x^3 + 3x^2 + x + 3)(2x - 1)$

For the polynomial $x^3 + 3x^2 + x + 3$ we proceed in a similar way and we find that:

$x^3 + 3x^2 + x + 3 = (ex^2 + f)(x + 3)$

or $x^3 + 3x^2 + x + 3 = ex^3 + 3ex^2 + fx + 3f$, from which by identification it results:

$e = 1$ and $f = 1$, and the polynomial finally becomes:

$P(x) = 2x^4 + 5x^3 - x^2 + 5x - 3 = (x^2 + 1)(x + 3)(2x - 1)$

This polynomial has two conjugated imaginary roots equal to ± i and two real roots equal to -3 and 1/2 respectively.

Then for the imaginary roots we calculate $Q(i) = i^3 + 6i^2 + 11i - 4a = -i - 6 + 11i - 4a = 10i - 6 - 4a = 0$, from which it results $a = \frac{-6 + 10i}{4}$

Similarly $Q(-i) = (-i)^3 + 6(-i)^2 + 11(-i) - 4a = i - 6 - 11i - 4a = -6 - 10i - 4a = 0$ from which it results

$a = \frac{-6 - 10i}{4}$

For the real roots we calculate $Q(-3) = (-3)^3 + 6(-3)^2 + 11(-3) - 4a = -27 + 54 - 33 - 4a = -6 - 4a = 0$ from which $a = -\frac{3}{2}$

and in the same way $Q(1/2) = (1/2)^3 + 6(1/2)^2 + 11(1/2) - 4a =$ zero from which $a = \frac{57}{32}$

So the polynomial $x^3 + 6x^2 + 11x - 4a$ can be decomposed in three versions, namely:

- the first version: $Q(x) = (x - i)\,Q1(x)$ with the parameter $a = \frac{-6 + 10i}{2}$
- the second version: $Q(x) = (x + i)Q2(x)$ with the parameter $a = \frac{-6 + 10i}{4}$
- the third version: $Q(x) = (x + 3)Q3(x)$ with the parameter $a = -\frac{3}{2}$
- the fourth version: $Q(x) = (x - ½)Q4(x)$ with the parameter $a = \frac{57}{32}$

For the fraction $\frac{P(x)}{Q(x)}$ to be irreducible, we need:

the parameter a does not belong to the set $\left\{ -\frac{3}{2} \;;\; \frac{57}{32} \;;\; \frac{-6 + 10i}{4} \;;\; \frac{-6 - 10}{4} \right\}$

Author
Adrian Stanculescu, Engineer, Ph.D.

BIBLIOGRAPHY:

1. Georgiana Popescu, Ileana Dumitru: “Elemente de matematici superioare comentate si explicate” [Elements of higher mathematics discussed and explained], Editura PIM [PIM Publishing House], Iasi, 2009

2. Danut Dracea, Liliana Niculescu, Ion Patrascu, Dan Seclaman: “Matematica. Manual pentru clasa a XII-a M1” [Mathematics. Handbook for Grade 12 M1], Editura Cardinal [Cardinal Publishing House], Ministry of Education, Reasearch and Youth, 2007.