A FEW CONSIDERATIONS REGARDING MATRICES

I. This material refers to mathematical methods designed for facilitating calculations in matrix operations.

In this case, we know the operations of multiplying two square matrices of configuration (m, m) with m rows and m columns of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \text{x} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix}$$

The result of the multiplication will also be a square matrix of type (m, m) with the following configuration:

$$\begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} & \dots & a_{11}\, b_{12} + a_{12}\, b_{22} + & \dots & a_{11} b_{1m} + \cdots \\ a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23} b_{31} & \dots & a_{21}\, b_{12} + a_{22}\, b_{22} + & \dots & a_{21} b_{1m} + \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}\, b_{11} + a_{m2}\, b_{21} + a_{m3} b_{31} & \dots & a_{m1}\, b_{12} + a_{m2}\, b_{22} + & \dots & a_{m1} b_{1m} + \cdots \end{pmatrix}$$

Thus the method of multiplication of two matrices is pretty simple and is made by a classical cross scheme.

The problem becomes complicated is we must multiply two matrices which do not appear in the configuration above. So, if for instance we must perform the following multiplication: one (m, n) matrix x one (n, n) matrix as in the following example where m = 2 and n =3:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \text{x} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

The multiplication result will be an (m, n) matrix, namely:

$$\begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} & a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13}\, b_{32} & a_{11}\, b_{13} + a_{12}\, b_{23} + a_{13}\, b_{33} \\ a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31} & a_{21}\, b_{12} + a_{22}\, b_{22} + a_{23}\, b_{32} & a_{21}\, b_{13} + a_{22}\, b_{23} + a_{23}\, b_{33} \end{pmatrix}$$

Below we propose an easy method allowing the multiplication of two matrices between them, regardless of their configuration.

Before proceeding to the development of the method itself, we will define the group (**G**,$*$) governed by a law of the form: $\mathbf{G} * \mathbf{G} \to \mathbf{G}$ $\quad$ $\mathbf{(X, Y)} \to \mathbf{X} * \mathbf{Y}$, and where **X** is an element of the multiplying matrices, and **Y** is a symbol element with no significant value.

So we can write that: **X * Y = Y * X = Y**

The elements are located in the matrices multiplying so as to fill the rows and columns of these matrices resulting in the final two square matrices.

Thus the two square matrices must be multiplied, each operation of multiplication of these two matrices sis performed after the classical method and if ℓ represents a matrix term, the resulting multiplication operations $\ell * \mathbf{Y}$ has no value therefore it is eliminated from the result at the end of the multiplication operation.

We will illustrate this method below, namely:

We must multiply two matrices as below:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \text{x} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

We complete the last row with **Y** symbols thus forming a square matrix of the type (m, m). Consequently, we must multiply two square matrices between them, which can be multiplied after the classical method.

The following matrix results:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{Y} \end{pmatrix} \text{x} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

From which we eliminate the **Y** symbols and we obtain the resulting matrix:

$$\begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} & a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13}\, b_{32} & a_{11}\, b_{13} + a_{12}\, b_{23} + a_{13}\, b_{33} \\ a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31} & a_{21}\, b_{12} + a_{22}\, b_{22} + a_{23}\, b_{32} & a_{21}\, b_{13} + a_{22}\, b_{23} + a_{23}\, b_{33} \\ \underbrace{\boldsymbol{Y}\, b_{11} + \boldsymbol{Y}\, b_{21} + \boldsymbol{Y}\, b_{31}} & \underbrace{\boldsymbol{Y}\, b_{12} + \boldsymbol{Y}\, b_{22} + \boldsymbol{Y}\, b_{32}} & \underbrace{\boldsymbol{Y}\, b_{13} + \boldsymbol{Y}\, b_{23} + \boldsymbol{Y}\, b_{33}} \end{pmatrix}$$

No value or significance	No value or significance	No value or significance

which is reduced to the following result:

$$\begin{pmatrix} a_{11}\, b_{11} + a_{12}\, b_{21} + a_{13}\, b_{31} & a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13}\, b_{32} & a_{11}\, b_{13} + a_{12}\, b_{23} + a_{13}\, b_{33} \\ a_{21}\, b_{11} + a_{22}\, b_{21} + a_{23}\, b_{31} & a_{21}\, b_{12} + a_{22}\, b_{22} + a_{23}\, b_{32} & a_{21}\, b_{13} + a_{22}\, b_{23} + a_{23}\, b_{33} \end{pmatrix}$$

Another example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \text{ x } \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

According to the method, it will be written as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{Y} \\ \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{Y} \end{pmatrix} \text{ x } \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Or:

$$\begin{pmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31} & a_{11}\,b_{12} + a_{12}\,b_{22} + a_{13}\,b_{32} & a_{11}\,b_{13} + a_{12}\,b_{23} + a_{13}\,b_{33} \\ \boldsymbol{Y}\,b_{11} + \boldsymbol{Y}\,b_{21} + \boldsymbol{Y}\,b_{31} & \boldsymbol{Y}\,b_{12} + \boldsymbol{Y}\,b_{22} + \boldsymbol{Y}\,b_{32} & \boldsymbol{Y}\,b_{13} + \boldsymbol{Y}\,b_{23} + \boldsymbol{Y}\,b_{33} \\ \boldsymbol{Y}\,b_{11} + \boldsymbol{Y}\,b_{21} + Y\,b_{31} & \boldsymbol{Y}\,b_{12} + Y\,b_{22} + Y\,b_{32} & \boldsymbol{Y}\,b_{13} + \boldsymbol{Y}\,b_{23} + \boldsymbol{Y}\,b_{33} \end{pmatrix}$$

From which according to the method, by eliminating the terms containing the $\mathbf{Y}$ symbol, it results:

$$\begin{pmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} + a_{13}\,b_{31} & a_{11}\,b_{12} + a_{12}\,b_{22} + a_{13}\,b_{32} & a_{11}\,b_{13} + a_{12}\,b_{23} + a_{13}\,b_{33} \end{pmatrix}$$

Through this system, by turning any matrix into a square matrix, we can perform a multiplication operation very easily.

Example: Let us suppose we must perform the following operation

$$\begin{pmatrix} 1 & 3 & 4 \end{pmatrix} \text{ x } \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

Following the proposed procedure, the multiplication becomes:

$$\begin{pmatrix} 1 & 3 & 4 \\ \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{Y} \\ \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{Y} \end{pmatrix} \text{ x } \begin{pmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ -2 & 0 & 1 \end{pmatrix}, \text{ namely}$$

$$\begin{pmatrix} 1*(-1)+3*3+4*(-2) & 1*2+3*1+4*0 & 1*0+3*1+4*1 \\ \boldsymbol{Y}*(-1)+\boldsymbol{Y}*3+\boldsymbol{Y}*(-2) & \boldsymbol{Y}*2+\boldsymbol{Y}*1+\boldsymbol{Y}*0 & \boldsymbol{Y}*0+\boldsymbol{Y}*1+\boldsymbol{Y}*1 \\ \boldsymbol{Y}*(-1)+\boldsymbol{Y}*3+\boldsymbol{Y}*(-2) & \boldsymbol{Y}*2+\boldsymbol{Y}*1+\boldsymbol{Y}*0 & \boldsymbol{Y}*0+\boldsymbol{Y}*1+\boldsymbol{Y}*1 \end{pmatrix}$$

ignoring the terms containing **Y** we obtain:

$$\begin{pmatrix} 1*(-1)+3*3+4*(-2) & 1*2+3*1+4*0 & 1*0+3*1+4*1 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 7 \end{pmatrix}$$

II. Another application consists of making a smoother invertible matrix. So, after the classical method, considering a matrix of the type:

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

And wishing to obtain the inverse matrix M^{-1}, we first create the transposed matrix M^{T}, that looks like this:

$$M^{T} = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix}$$

IIn the next step, from the transposed matrix we form the adjugate matrix that looks like this:

$$M^{*} = \begin{pmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nn} \end{pmatrix}$$

In this matrix, the minors d_{ij} represent an algebraic complement obtained by cutting the row *i* and the column *j* of the transposed matrix and these minors multiply by $(-1)^{(i+j)}$.

For example, d_{23} is obtained by cutting the row 2 and column 3 of the initial matrix, obtaining a minor which multiplies by $(-1)^{(2+3)}$, and the inverse matrix is equal to:

$$M^{-1} = \frac{1}{\Delta} M^*$$

where Δ represents the determinant of the initial matrix.

In other words, in order to obtain the adjugate matrix, first we create the transposed matrix, then we extract the minors d_{ij} from this equation and then we form the algebraic complements by multiplying the minors d_{ij} by $(-1)^{(i+j)}$.

Through this article we propose an easier method to obtain the adjugate matrix directly from the initial matrix without the need to create the transposed matrix.

In this situation, this matrix will look as follows:

$$M^* = \begin{pmatrix} d_{11} & d_{21} & d_{31} & \cdots & d_{n1} \\ d_{12} & d_{22} & d_{32} & \cdots & d_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{1n} & d_{2n} & d_{3n} & \cdots & d_{nn} \end{pmatrix}$$

where d_{ij} represents the minor obtained by cutting the row *i* and the column *j* multiplied by $(-1)^{(i+j)}$ the operation being performed directly from the initial matrix.

We will illustrate the proposed method by an example. Let it be the matrix:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 4 \\ -3 & -4 & 1 \end{pmatrix}$$

We are asked to calculate its inverse. After the classical method, first we calculate the transposed matrix by reversing the rows with the columns.

$$A^T = \begin{pmatrix} 1 & 3 & -3 \\ -2 & -1 & -4 \\ 0 & 4 & 1 \end{pmatrix}$$

Then we calculate the adjugate matrix:

$$A^* = \begin{pmatrix} 15 & 2 & -8 \\ -15 & 1 & -4 \\ -15 & 10 & 5 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

And the inverse matrix is equal to:

$$A^{-1} = \frac{1}{\Delta} A^{*} = \frac{1}{45} * \begin{pmatrix} 15 & 2 & -8 \\ -15 & 1 & -4 \\ -15 & 10 & 5 \end{pmatrix}$$

After the proposed method, the steps to obtain the inverse matrix are the same with the difference that the transposed matrix is no longer calculated and the adjugate matrix is obtained directly from the initial matrix, namely:

$$A^{*} = \begin{pmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{pmatrix}$$

The significance of the algebraic complements d_{ij} being the same, namely:

$$A^{*} = \begin{pmatrix} 15 & 2 & -8 \\ -15 & 1 & -4 \\ -15 & 10 & 5 \end{pmatrix}$$

for example, the algebraic complement d_{21} being obtained by cutting row 2 and column 1 of the initial matrix, the resulting determinant finally multiplying by $(-1)^{(2+1)}$ and the inverse matrix will be equal to:

$$A^{-1} = \frac{1}{45} * \begin{pmatrix} 15 & 2 & -8 \\ -15 & 1 & -4 \\ -15 & 10 & 5 \end{pmatrix}$$

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